ON DITROIDS AND BISUBMODULAR SYSTEMS

by

RANGAN K. GUHA

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

SEPTEMBER, 1994

ON DITROIDS AND BISUBMODULAR SYSTEMS

A Thesis Submitted
in Partial Fulfilment of the Requirements
for the Degree of
DOCTOR OF PHILOSOPHY

By
RANGAN K. GUHA

to the

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY KANPUR

SEPTEMBER, 1994

Certificate

It is certified that the work contained in this thesis entitled "**On Ditroids and Bisubmodular Systems**", by *Rangan K. Guha*, has been carried out under my supervision and that this work has not been submitted elsewhere for a degree or diploma.

Prabha Sharma

(DR. PRABHA SHARMA.)
Professor,
Department of Mathematics,
Indian Institute of Technology,
Kanpur-208016, INDIA

September, 1994

ABSTRACT

Theory of matroids and submodular functions has played a significant role in the development of theory of Combinatorial Optimization. To cover a variety of combinatorial optimization problems which failed to have the matroidal or submodular structures, generalised matroids, pseudomatroids, ditroids and bisubmodular systems got introduced. In this thesis an attempt has been made to extend the theory and applications of ditroids and bisubmodular systems further. Polyhedral structures of a bisubmodular polyhedron have been studied in details and it has been shown that for all special cases of the bisubmodular system the corresponding results can be obtained from these. All integer vectors of a bisubmodular polyhedra with $(0, \pm 1)$ extreme points have been shown to satisfy the 2-augmentation property and we call them 'Greedy System'. Ditroids are greedy systems. The membership problem for the bisubmodular polyhedron has been shown to be equivalent to a bisubmodular function minimization and it is suggested that an existing pseudo polynomial algorithm for submodular polyhedra can be extended to solving the membership problem for the bisubmodular polyhedron. Lastly a few directions for future work have been given.

To

My Parents -

my source of inspiration.

Acknowledgements

It was a great honour to work with Prof.(Mrs.) Prabha Sharma in the course of my Ph.D dissertation. Her devotion to work, sharp analytical ability, positive thinking coupled with her complete authority on the subject have culminated in this thesis. She has always given me free hand to develop my own ideas and has always helped to clear my concepts. I owe her a lot for the present level of my understanding of the subject.

I gratefully acknowledge the help, enthusiasm and critical comments received from Drs. P.C. Das, Debasis Kundu, P. Shunmugaraj, Santokh Singh. Useful discussions with them helped to remove many a bottleneck. I sincerely thank all the faculty member and the non-teaching staff of Mathematics Department for their help.

I am thankful to Dr. S.N. Kabadi, faculty, University of New Brunswick, Canada, for his helpful discussions and suggestions during his visit at I.I.T, Kanpur.

Looking down the memory lane, I recall many personalities like Drs. S.K. Gupta, T.K. Mukherjee, S. Nandy, Messrs R.S. Ghosh, R.N. Ghosh and C.S. Roychowdhury who have encouraged me in many ways towards the completion of the thesis. Thank you sirs, for your constant inspiration which will be etched in my memory for ever.

It is a pleasure to acknowledge my colleagues Drs. R.K. Pandey, A.K. Singh, R.K. Bawa, Messrs Sokkolingam, Amit, Rao, Panda, Balram, Dipak, Joydip, Bera, Mmes. Pratibha, Nisha, Ratna and Sharmistha for their help and company during my stay in the campus. The credit of thesis typing goes to a large extent to Dipak and Joydip, who introduced me to the world of LATEX. After staying in Hall-IV for a long period, it is difficult task to list down the names of the friends who have made my stay a memorable one. It will be difficult to forget friends like Baba, Mogambo, Mada, Mama, Frenchi, Kak, Bogada, Dada, Phoolda, Rangada, Fosbabu, Buddhababu, Roybabu, Jalbabu, Khuski, Damra, Machao, Netyakoli, Mota-Pani, Charpoka Singh and many others who did not figure in this list. It was a great oppurtunity to be associated with Ranjana, Rupa and Tenpirani, who helped through their spirited nature in enjoying campus life.

The pains and triumphs that I shared with all the members of the Institute Football team which I had the honour to represent during the last five years add to the nostalgia that is associated with my stay here. Thank you all for the love and respect you have showered on me.

Thanks are also due to Mr. Narayan and Mr. Brij Lal for the active 'Tea' support from them.

The silent and constant encouragement and support from my family members had been a source of inspiration in completing this work.

Rangan K. Guha.

CONTENTS

Chapter 1

INTRODUCTION

1.1 Introduction and Literature Survey

It has been established beyond doubt that the theory of matroids and submodular functions has played a significant role in the development of theory of Combinatorial Optimization.

Matroids were first introduced in 1930s but their applications to combinatorial optimization began with Rado's and Edmond's work, much later.

By generalising the matroid axioms, it was possible to introduce subset systems which had the greediness property, but were not matroids. These are delta-matroids [6] or pseudo-matroids [7], generalised polymatroids [22], symmetric matroids etc. Oriented matroids [3] gave a richer abstraction of vector spaces over ordered fields, than matroids.

Edmonds in his pioneering paper in 1970 [15] introduced submodular functions and the submodular polyhedra. In particular the matroid polyhedron is a submodular polyhedron and the matroid rank function, a submodular function on the subsets of the ground set E of the matroid. He showed that a submodular polyhedron has the total dual integrality property and in case the submodular function defining the polyhedron is integer valued then every extreme point of the polyhedron is an integer valued vector. Also the greedy algorithm

solves the LPP associated with a submodular polyhedron.

Further research focused on finding the largest class of polyhedra which have the total dual integrality property and also the greedy property in a generalised sense. Some of these results are in [33], [37], [43], [47] and [51].

The membership problem for a submodular polyhedra is, given a vector $x \in \Re^n$, determine whether it belongs to a submodular polyhedron $\mathcal{P}_f$, and if not then, find the maximum violated inequality of $\mathcal{P}_f$, and the submodular function minimisation problem is, finding a subset X of E for which $f(X)$ has the smallest value, are the other aspects of submodular theory which have been studied in [9], [10], [13], [37] and [50]. It has been shown that the membership problem for a submodular polyhedron and a submodular function minimisation are equivalent problems and polynomially solvable.

Another aspect of submodularity theory which has been extensively studied is the polyhedral structure of the submodular polyhedron. Faces, facets and extreme points on the submodular polyhedron have been defined and necessary and sufficient conditions for two extreme points to be adjacent have been given in [26], [31], [52] and [54].

Polyhedral structures of a generalised polymatroid have been analysed by Frank and Tardos in [24]. Polyhedral structures of a perfectly matchable subgraph polytope, which is a pseudomatroid polyhedron, have been studied in [1] and [13].

In the direction towards generalising these results further, comes the theory of bisubmodular functions, bisubmodular polyhedra and jump systems.

Given a finite ground set E, and a collection of disjoint pair of subsets $D(E) = \{(A, B) : A, B \subseteq E \text{ and } A \cap B = \phi\}$, a function $f : D(E) \longrightarrow \Re \cup \{\infty\}$, is called bisubmodular if it satisfies for (A, B), $(C, D) \in D(E)$,

$$f(A, B) + f(C, D) \geq f(A \cup C / B \cup D, B \cup D / A \cup C) + f(A \cap C, B \cap D).$$

Rank functions of a pseudomatroid and g-polymatroid are bisubmodular functions. The polyhedra associated with a bisubmodular function is a bisubmodular polyhedron.

Base polyhedra, the pseudomatroid polyhedra and the generalised polymatroid are all bisubmodular polyhedra. Various aspects of these bisubmodular polyhedra and the bisubmodular functions have been studied in [6], [14], [33], [42] and [43].

In his paper, 'directed submodularity, ditroids and directed submodular flows' [43], Qi generalised the concept of matroids to directed subsets i.e. to $D(E)$, as defined earlier, and called them ditroids. Because the set relations and operations such as union and intersection are not the same for directed sets as for undirected sets, ditroids are a genuine generalisation of matroids. A fundamental difference between the two being that the maximal independent sets of a ditroid need not have the same cardinality. Qi also proved that the rank function of a ditroid is a bisubmodular function.

1.2 Summary of the Thesis

We now give a chapter wise summary of the thesis.

In chapter two an attempt is made to see how far the results valid for matroids continue to be valid for ditroids. Some of the augmenting axioms for matroids have been extended to ditroids, and for two independent disets X, Y of a ditroid, such that $|Y| = |X| + 1$, we show that an augmenting path exists for X, with respect to Y. Some of the properties of the rank functions of a matroid have also been extended to the rank function of a ditroid.

Circuit axioms for ditroids have been obtained and the dual of a ditroid has also been defined.

Different operations on ditroids such as deletion, contraction, reflection and projection have been defined and it is shown that all these operations yield new ditroids.

Composition of two ditroids is shown to yield a new ditroid and we also show that any ditroid can be decomposed into two matroids.

Relationships between ditroids and other subset systems such as symmetric matroids

and oriented matroids have been established. We show that oriented matroids are self dual ditroids. Conforti and Laurent's [8] characterization of matroidal system of inequalities has been extended to characterization of ditroidal system of inequalities.

In chapter three, we are able to add another polyhedron, the degree sequence polytope to the list of bisubmodular polyhedron. We show that a bisubmodular polyhedron $\mathcal{P}_f$ is non-empty if and only if $f(A,B)+f(B,A)\geq 0$, for all $(A,B)\in D(E)$, and that $f(\phi,\phi)=0$ is a sufficient condition for $\mathcal{P}_f$ to be non-empty.

$f\,:\,D(E)\longrightarrow\Re$, is a finite valued bisubmodular function if,

$$f(A,B)<\infty,\text{ for all }(A,B)\in D(E).$$

We define the generalised greedy algorithm (gga) for an LPP, associated with $\mathcal{P}_f$ for f finite valued and show that the gga solves the associated LPP.

Solutions of LPP associated with the pseudomatroid polyhedra, g-polymatroid and the degree sequence polytope can be obtained by this generalised greedy algorithm.

By the gga it can be shown that if for some $(Y_1,Y_2)\in D(E)$,

$$g(Y_1,Y_2)=\max\{x(Y_1,Y_2),\ x\in\mathcal{P}_f\}\ \text{ and }\ g(\phi,\phi)=0$$

then $g(Y_1,Y_2)=f(Y_1,Y_2)$, whenever f is finite valued.

Total dual integrality of the bisubmodular polyhedron $\mathcal{P}_f$ has been established via the gga and in case f is integer valued, this implies that the LPP solutions associated with $\mathcal{P}_f$ will yield integer solutions. Qi [43] proves the TDI-ness of $\mathcal{P}_f$ by a different method.

In section 3.5.2, we define a Greedy System as the collection of all $(0,\pm1)$ vectors in a bisubmodular polyhedron with $(0,\pm1)$ extreme points. We show that greedy systems satisfy a 2-augmentation property - a generalisation of the 1-augmentation property of matroids. Greedy systems which are closed with respect to set inclusion, in the directed sense, are nothing but ditroids. We also prove the converse of this statement i.e., every ditroid is a greedy system.

Finally in section 3.5.3, we consider different ways of obtaining subsets of a greedy systems which also satisfy 2-SA.

In the fourth and last chapter, we focus on the polyhedral structure of the bisubmodular polyhedron. Faces and facets of $\mathcal{P}_f$ are defined and a necessary and sufficient condition for a diset inequality $x(A,B) \leq f(A,B)$ to represent a facet of $\mathcal{P}_f$, is given in terms of non-separability of (A,B) with respect to f.

It is further shown that a facet of $\mathcal{P}_f$ is again a bisubmodular polyhedron, and a facet of a ditroid polyhedron is again a ditroid polyhedron and a facet of a pseudomatroid polyhedron is also a pseudomatroid polyhedron. Facets of the degree sequence polytope, in our sense, reduce to the facets obtained by Peled and Srinivasan in [41].

Extreme points of $\mathcal{P}_f$ have been defined and a polynomial time algorithm to check if a given vector $x \in \Re^n$ is an extreme point of $\mathcal{P}_f$ is given. This algorithm yields the non-cancelling tight sets with respect to x and in case the number of independent tight sets is less than $\dim(\mathcal{P}_f)$, x is not an extreme point of $\mathcal{P}_f$.

Necessary and sufficient conditions for two extreme points to be adjacent on $\mathcal{P}_f$ have been given. Specialised characterizations of extreme points and adjacency for the ditroid polyhedron and pseudomatroid polyhedron have been obtained. Our results on characterization of extreme points and adjacency for g-polymatroid and the degree sequence polytope tally with the results obtained in [24] and [41] respectively.

Throughout this chapter we point out that because the set of tight sets with respect to any $x \in \mathcal{P}_f$ may form more than one distributive lattice, characterization of extreme points and adjacency is not very straight forward as it is for the submodular polyhedra.

Finally we state and prove the min-max theorem for the bisubmodular polyhedron and then go on to show that the membership problem for $\mathcal{P}_f$ and a bisubmodular function minimization problem are equivalent. The existing algorithm in [2] for testing membership in a polymatroid could not be generalised to $\mathcal{P}_f$, since it seemed equally difficult to choose a starting feasible solution for the algorithm. However, we do point out that the algorithm

in [50] should work for membership problem for $\mathcal{P}_f$ and hence for bisubmodular function minimization, but it will be a pseudo-polynomial algorithm.

1.3 Preliminaries

In this section we define '**what is a directed set**' and the various relations and operations among the directed subsets. Most of these definitions were given by Qi [43], [44].

Definition 1.3.1

(i) **Directed Set.** Let E be a finite set, with cardinality of $E = |E| = n$. A directed set (or diset) $X = (X_1, X_2)$ of E is an ordered pair of disjoint subsets of E i.e $X_1 \subseteq E$ and $X_2 \subseteq E$ and $X_1 \cap X_2 = \phi$. Also we identify a diset $X = (X_1, X_2)$ of E with an n-dimensional $\{0, \pm 1\}$ vector by letting for any $e \in E$

$$X(e) = \begin{cases} 1 & \text{if } e \in X_1 \\ -1 & \text{if } e \in X_2 \\ 0 & \text{otherwise.} \end{cases}$$

Let $D(E)$ denote the collection of all directed subsets of E. We will also regard the elements of $D(E)$ as the n-dimensional $\{0, \pm 1\}$ characteristic vectors, defined above.

Suppose $X = (X_1, X_2)$ and $Y = (Y_1, Y_2) \in D(E)$ and $x \in R^n$. Since X and Y can be regarded as $\{0, \pm 1\}$ vectors, we define,

(ii) $X = -Y$ if $X_1 = Y_2$ and $X_2 = Y_1$.

(iii) $X \leq Y$ if $X_1 \subseteq Y_1$ and $X_2 \supseteq Y_2$.

(iv) Xu or $u(X) = \sum_{e \in E} X(e)u(e)$.

By (iii), $D(E)$ forms a **lattice** with the lattice operations join and meet defined as

(v) $X \wedge Y = (X_1 \cap Y_1, X_2 \cup Y_2)$.

(vi) $X \vee Y = (X_1 \cup Y_1, X_2 \cap Y_2)$.

We say,

(vii) $\quad X \subset Y$ if and only if $X_1 \subset Y_1$ and $X_2 \subset Y_2$,

and

(viii) $\quad X \cap Y = (X_1 \cap Y_1, X_2 \cap Y_2)$ that is,

$$(X \cap Y)(e) = \begin{cases} 1 & \text{if } X(e) = Y(e) = 1 \\ -1 & \text{if } X(e) = Y(e) = -1 \\ 0 & \text{otherwise.} \end{cases}$$

(ix) $\quad X \cup Y = ((X_1 \cup Y_1)/(X_2 \cup Y_2), (X_2 \cup Y_2)/(X_1 \cup Y_1))$ that is,

$$(X \cup Y)(e) = \begin{cases} 1 & \text{if } X(e) + Y(e) > 0 \\ -1 & \text{if } X(e) + Y(e) < 0 \\ 0 & \text{if } X(e) + Y(e) = 0 \end{cases}$$

(x) $\quad X/Y = (X_1/(Y_1 \cup Y_2), X_2/(Y_1 \cup Y_2))$ that is,

$$(X/Y)(e) = \begin{cases} 1 & \text{if } X(e) = 1 \ \& \ Y(e) = 0 \\ -1 & \text{if } X(e) = -1 \ \& \ Y(e) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Other definitions of X/Y are possible, but we have found this definition quite useful.

(xi) $\quad$ We say X and Y are **intersecting** if $X \not\subset Y$, $Y \not\subset X$ and $X \cap Y \neq \phi$.

(xii) $\quad X$ and Y are said to be **negatively intersecting**, if X and $-Y$ are intersecting. Where $-Y = (Y_2, Y_1)$.

(xiii) $\quad X$ and Y are said to be **algebraically intersecting**, if they are intersecting or negatively intersecting.

(xiv) $\quad X$ and Y are **non-cancelling**, if and only if $X \cap (-Y) = \phi = Y \cap (-X)$. Otherwise, X and Y are **cancelling**.

(xv) $\quad \overline{X} = \text{absolute } (X) = X_1 \cup X_2$, so we have,

$|X| = |\overline{X}| = |X_1 \cup X_2| = |X_1| + |X_2|$.

(xvi) We say **invert** $F \subseteq E$ in X, if we change the sign of $X(i)$ in X to $-X(i)$ for all $i \in F$.

(xvii) We define symmetric difference between two disets, X and Y as follows: $X \Delta Y = (X/Y) \cup (Y/X)$.

(xviii) Let $\mathcal{T}$, denote some collection of disets of E. $\mathcal{T}$ is said to be an **intersecting family** if $X, Y \in \mathcal{T}$ and $X \cap Y \neq \phi$ implies $X \cap Y \in \mathcal{T}$ and $X \cup Y \in \mathcal{T}$.

We will now give some properties of disets, with their proofs whenever necessary. Suppose $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$ and $Z = (Z_1, Z_2)$ are three disets of E. Then the following properties are true.

Property 1.3.1

(P1) if $X \subseteq Y \subseteq Z$ then $X \subseteq Z$.

(P2) $X \subseteq Y \subseteq X \Rightarrow X = Y$.

(P3) $X \cap Y = Y \cap X$.

(P4) $X \cup Y = Y \cup X$.

(P5) $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$.

(P6) $X \cap Y \subseteq X \cup Y$.

(P7) $(-X) \cap (-Y) = -(X \cap Y)$

(P8) $(-X) \cup (-Y) = -(X \cup Y)$.

(P9) $X \cap (Y \cap Z) = (X \cap Y) \cap Z$

(P10) $X \cup Y = \phi$ iff $X = -Y$.

(P11) if $X \subseteq Y$ then $X \cap Z \subseteq Y \cap Z$.

(P12) if $X \subseteq Y$ and $X \subseteq Z$ then $X \subseteq Y \cap Z$.

(P13) if $X \subseteq Y$ then $X \cap (-Y) = (-X) \cap Y = \phi$.

(P14) $(X \cup Y) \cap Z = \{(X \cup Z) \cap (Y \cup Z)\} \cap \{(X \cap Z) \cup (Y \cap Z)\}$.

(P15) $(X \cap Y) \cup Z = \{(X \cup Z) \cap (Y \cup Z)\} \cup \{(X \cap Z) \cup (Y \cap Z)\}$.

Proof of (P14). In case X and Y are cancelling

$$(X \cup Y) \cap Z \subseteq (X \cap Z) \cup (Y \cap Z). \tag{1.3.1}$$

Since the cancelling elements of X and Y, which are in Z, are not present in the left hand side of (1.3.1) but will be present in the right hand side of (1.3.1). Moreover these cancelling elements do not belongs to $(X \cup Z) \cap (Y \cup Z)$. And for the remaining elements of X and Y, if $e \in (X \cup Y) \cap Z$ then also $e \in \{(X \cup Z) \cap (Y \cup Z)\} \cap \{(X \cap Z) \cup (Y \cap Z)\}$, hence

$$(X \cup Y) \cap Z = \{(X \cup Z) \cap (Y \cup Z)\} \cap \{(X \cap Z) \cup (Y \cap Z)\}.$$

When X and Y are non-cancelling,

$$(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z),$$

and

$$(X \cap Z) \cup (Y \cap Z) \subseteq (X \cup Z) \cap (Y \cup Z),$$

this implies

$$(X \cup Y) \cap Z = \{(X \cup Z) \cap (Y \cup Z)\} \cap \{(X \cap Z) \cup (Y \cap Z)\}. \ \square$$

Proof of (P15). In case X and Y are cancelling

$$(X \cup Z) \cap (Y \cup Z) \subseteq (X \cap Y) \cup Z. \tag{1.3.2}$$

Since the cancelling elements of X and Y, which are in Z, are not present in the left hand side of (1.3.2) but they will be present in the right hand side of (1.3.2). Moreover these cancelling elements belong to $(X \cap Z) \cup (Y \cap Z)$. And for remaining elements of X and Y, if $e \in (X \cap Y) \cup Z$ then also $e \in \{(X \cup Z) \cap (Y \cup Z)\} \cup \{(X \cap Z) \cup (Y \cap Z)\}$, hence

$$(X \cap Y) \cup Z = \{(X \cup Z) \cap (Y \cup Z)\} \cup \{(X \cap Z) \cup (Y \cap Z)\}.$$

When X and Y are non-cancelling,

$$(X \cup Z) \cap (Y \cup Z) = (X \cap Y) \cup Z,$$

and

$$(X \cap Z) \cup (Y \cap Z) \subseteq \{(X \cup Z) \cap (Y \cup Z)\},$$

hence

$$(X \cap Y) \cup Z = \{(X \cup Z) \cap (Y \cup Z)\} \cup \{(X \cap Z) \cup (Y \cap Z)\}. \ \square$$

However the following are not true in general:

(1) $X \subseteq X \cup Y$ or $Y \subseteq X \cup Y$. (2) $(X \cup Y) \cup Z = X \cup (Y \cup Z)$

(3) $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$. (4) $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$.

A counter example is:

$$X = (e, \phi), \ Y = (\phi, e) \text{ and } Z = (e, \phi).$$

Definition 1.3.2

$f : D(E) \to \Re$, is a **bisetfunction** on $D(E)$.

(i) f is said to be **non-decreasing** if $f(Y) \leq f(X)$ for all $Y \subseteq X$, X, $Y \in D(E)$.

(ii) We say that f is a **Fujishige bisubmodular** [26], [28] function if for any $X = (X_1, X_2)$ and $Y = (Y_1, Y_2) \in D(E)$,

$$f(X) + f(Y) \geq f(X \wedge Y) + f(X \vee Y),$$

i.e

$$f(X_1, X_2) + f(Y_1, Y_2) \geq f(X_1 \cap Y_1, X_2 \cup Y_2) + f(X_1 \cup Y_1, X_2 \cap Y_2). \tag{1.3.3}$$

(iii) f is said to be **bisubmodular** (or, directed submodular [43]) if for any $X = (X_1, X_2)$ and $Y = (Y_1, Y_2) \in D(E)$,

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y),$$

i.e

$$\begin{aligned} f(X_1, X_2) + f(Y_1, Y_2) \;\; &\geq \;\; f(X_1 \cap Y_1, X_2 \cap Y_2) \\ &+ \;\; f((X_1 \cup Y_1)/(X_2 \cup Y_2), (X_2 \cup Y_2)/(X_1 \cup Y_1)). \end{aligned} \tag{1.3.4}$$

It has been shown that the generalized submodular function [22], rank functions of a pseudomatroid [7], delta-matroid [5] and ditroid [43] are all examples of bisubmodular functions.

For any $e \in E$ and $X = (X_1, X_2)$ such that $e \notin X$, we write, $X + e$ in place of $X \cup (e, \phi)$ or $X \cup (\phi, e)$. It will always be clear from the context, whether e is a forward or a backward element of $X + e$.

Define **incremental function** of a biset function f at $S \in D(E)$ as follows :

$$f_j(S) = f(S + j) - f(S). \tag{1.3.5}$$

This definition generalises the lower and upper incremental functions of f, given by Qi, [44].

We now give some properties of a bisubmodular function. These properties are extensions of the properties of submodular functions. The list here is by no means exhaustive. Only those results which are to be used in developing the theory further are given.

Property 1.3.2 Let $f : D(E) \longrightarrow \Re$ be a non-decreasing biset function. f is bisubmodular if and only if its incremental function is non-increasing, i.e, if and only if,

$$f_j(S) = f(S+j) - f(S) \geq f(S+j+k) - f(S+k) = f_j(S+k), \tag{1.3.6}$$

$$\text{for all } j,k \in E/\overline{S}\ \ j \neq k.$$

As mentioned earlier $(S + j)$ means that either j has been added as a forward or as a backward element.

Proof. f bisubmodular, implies

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B), \text{ for all } A,\ B \in D(E).$$

Putting $A = S + j$, where j, may be forward or backward element of A, and $B = S + k$, where k is a forward or backward element of B.
We have,

$$f(S+j) + f(S+k) \geq f(S \cup \{j,k\}) + f(S),$$

or

$$f(S+j) - f(S) \geq f(S \cup \{j,k\}) - f(S+k).$$

Conversely, let (1.3.6) be true, that is,

$$f(S+j) - f(S) \geq f(S \cup \{j,k\}) - f(S+k)$$

hold. Let $S = A \cap B$ for some $A,\ B \in D(E)$ and $A,\ B$ are non-cancelling,let, $A/B = \{j_1, j_2, \cdots j_r\}$ and $B/A = \{k_1, k_2, \cdots k_s\}$, forward and backward elements will be known from the context. Then,

$$S \cup \{k_1, k_2, \ldots, k_s\} \cup \{j_1, j_2, \ldots, j_r\} = A \cup B.$$

Now,

$$\begin{aligned} f(B) \ - \ & f(A \cap B) \\ = \ & \sum_{t=1}^{s} [f(S \cup \{k_1, k_2, \cdots, k_t\}) - f(S \cup \{k_1, k_2, \cdots, k_{t-1}\})] \\ = \ & \sum_{t=1}^{s} [f((S \cup \{k_1, k_2, \cdots, k_{t-1}\}) + k_t) - f(S \cup \{k_1, k_2, \cdots, k_{t-1}\})] \end{aligned}$$

$$\geq \sum_{t=1}^{s}[f(S \cup \{k_1, k_2, \cdots, k_{t-1}\} + \{k_t, j_1\}) - f(S \cup \{k_1, k_2, \cdots, k_{t-1}\} + j_1)]$$

$$\vdots \qquad \text{using (1.3.6) for } j_1$$

$$\geq \sum_{t=1}^{s}[\, f(S \cup \{k_1, k_2, \cdots k_t\} \cup \{j_1, j_2, \cdots, j_r\}) - f(S \cup \{k_1, k_2, \cdots, k_{t-1}\} \cup \{j_1, j_2, \cdots, j_r\})]$$

$$\text{using (1.3.6) for } j_1, j_2, \ldots, j_r$$

$$= f(A \cup B) - f(A).$$

Hence $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$.

Suppose $A = (A_1, A_2)$, $B = (B_1, B_2)$ are cancelling. Define

$$A' = (A_1/A_1 \cap B_2, A_2/A_2 \cap B_1) \text{ and } B' = (B_1/A_2 \cap B_1, B_2/A_1 \cap B_2).$$

Then A' and B' are non-cancelling and $A \cup B = A' \cup B'$ and $A \cap B = A' \cap B'$. Also f is an increasing function. So

$$f(A) + f(B) \geq f(A') + f(B') \geq f(A' \cup B') + f(A' \cap B') = f(A \cup B) + f(A \cap B). \quad \square$$

Property 1.3.3 f is bisubmodular and non-decreasing if and only if, for any two non-cancelling disets S, $T \in D(E)$

$$f(T) \leq f(S) + \sum_{j \in T/S} \{f(S + j) - f(S)\}. \tag{1.3.7}$$

Proof. Let f be bisubmodular and non-decreasing. Let $T/S = \{j_1, j_2, \cdots, j_r\}$, then,

$$\begin{aligned} f(T) \leq f(S \cup T) &= f(S) + \{f(S \cup T) - f(S)\} \\ &= f(S) + \sum_{t=1}^{r}[f(S \cup \{j_1, j_2, \cdots, j_t\}) - f(S \cup \{j_1, j_2, \cdots, j_{t-1}\})] \\ &\leq f(S) + \sum_{t=1}^{r}[f(S + j_t) - f(S)]. \end{aligned}$$

Where the first inequality holds, because f is non-decreasing, and the second inequality holds, because f is bisubmodular.

To prove the converse, let (1.3.7) be true, that is,

$$f(T) \leq f(S) + \sum_{j \in T/S} [f(S + j) - f(S)],$$

for any two non-cancelling disets $S,\ T \in D(E)$. Taking $T = S/\{j\}$, gives $f(S/\{j\}) \leq f(S)$, which implies f is non-decreasing.
Taking $T = S \cup \{j,\ k\}$, this gives

$$\begin{aligned} f(S \cup \{j,k\}) &\leq f(S) + f(S+j) - f(S) + f(S+k) - f(S) \\ \text{or,} \qquad f(S+j) - f(S) &\geq f(S \cup \{j,k\}) - f(S+k). \end{aligned}$$

Hence, f is bisubmodular, by property (1.3.2). □

Let $S^1, S^2, \cdots, S^n$, be directed subsets of E, such that

$$\phi = S^0 \subset S^1 \subset S^2 \subset \cdots \subset S^n.$$

Where, $\overline{S^n} = E$ and $S^i = \{e_1, e_2, \cdots, e_i\}$, considering their direction also.
Then $S^0, S^1, S^2, \cdots, S^n$ forms a **directed chain** in $D(E)$. We now prove the following two properties of the function f, with respect to the chain $S^0, S^1, S^2, \cdots, S^n$.

Property 1.3.4 If $A = (A_1, A_2) \in D(E)$ and j is the first index for which $e_j \in S^j$ and $e_j \notin (A_1, A_2)$ and $e_i \in (A_1, A_2)\ \forall\ i < j$, and also A is non-cancelling with S^j, then

$$f(A_1, A_2) + f(S^j) \geq f(S^{j-1}) + f(A_1, A_2) \cup S^j). \tag{1.3.8}$$

Proof. Since $(A_1, A_2) \cap S^j = S^{j-1}$, the proof follows from bisubmodularity of f. □

Property 1.3.5 If $A = (A_1, A_2) \in D(E)$ and j is the first index for which $e_j \in S^j$, but $A = (A_1, A_2)$ is cancelling with S^j with respect to e_j [i.e, if $e_j \in A_1$ then $e_j \in S^j_2$, or if, $e_j \in A_2$ then $e_j \in S^j_1$.], then

$$f(A_1, A_2) + f(S^j) \geq f(S^{j-1}) + f(A_1, A_2) \cup S^j).$$

Proof. Since $(A_1, A_2) \cap S^j = S^{j-1}$, the proof follows from bisubmodularity of f. □

Chapter 2

DITROIDS

?.1 Introduction

n this chapter we first describe in brief Qi.'s ([43], [44]) work on ditroids, who introduced hem first. In section two, we show that a ditroid can be split into two matroids and also escribe conditions under which the $(0, \pm 1)$ solutions to a system of linear inequalities will epresent a ditroid.

In section three we show the existence of an augmenting path P for any $X \in \mathcal{I}$ such that a $Y \in \mathcal{I}$ of higher cardinality exists, then $X \cup P \in \mathcal{I}$ and $|X \cup P| = |X| + 1$. In section our, we prove some more properties of rank functions of ditroids.

In section five, we define circuits and circuit axioms for ditroids and their properties. In ection six, well defined operations on matroids, like contraction, deletion and projection are stablished for ditroids.

And in section seven, we define the dual of a ditroid and its rank function, focus on perfect itroids and composition of two ditroids to construct new ditroids. Finally, in section eight, e establish relationship between ditroids and some other subset systems and show that iented matroids are self dual ditroids.

2.2 Definitions, Examples and Relation with Matroids

Definition 2.2.1 ([43], [44]).

(1) Let $\mathcal{I} \subseteq D(E)$. $D = (E, \mathcal{I})$ is said to be a **ditroid** if the family $\mathcal{I}$ satisfies the following axioms:

(D1) $\phi \in \mathcal{I}$;

(D2) if $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$;

(D3) if X and $Y \in \mathcal{I}$ are non-cancelling, with $|X| = |Y| + 1$, then there exists $e_i \in E$ and $Z \in \mathcal{I}$ such that
$Y(e_i) = 0$, $Z(e_i) = X(e_i) \neq 0$, and $Z(e_j) = Y(e_j)$ for all $j \neq i$.

Axiom (D3) is the directed version of the augmentation axiom for matroids.

(2) $I \in \mathcal{I}$ is an **independent** set of D. All other disets of $D(E)$ are **dependent** sets of D.

(3) A maximal independent diset of $\mathcal{I}$ is called a **base** of $D = (E, \mathcal{I})$.

(4) A minimal dependent diset of D, is said to be a **circuit** of D.

(5) The **rank function** of a ditroid $D = (E, \mathcal{I})$, is a set function, $h : D(E) \longrightarrow \Re_+$, defined by

$$h(X) = h(X_1, X_2) = \max\{|Y| \,:\, Y \subseteq X \text{ and } Y \in \mathcal{I}\} \text{ for } X \in D(E).$$

h is bisubmodular on $D(E)$ [43].

Following are some examples of ditroids [43], [44].

Example 2.2.1 Suppose that $M_1 = (E, \mathcal{J}_1)$ and $M_2 = (E, \mathcal{J}_2)$ are two matroids on E. Let

$$\mathcal{I} = \{(A_1, A_2) \in D(E) \,:\, A_1 \in \mathcal{J}_1,\, A_2 \in \mathcal{J}_2)\}.$$

Then $D = (E, \mathcal{I})$, is a ditroid.

Example 2.2.2 $G = (V, E)$, is a graph. A diset $(A_1, A_2) \in D(V)$, is said to be covered by a matching M, if A_1 and A_2 are covered by M and no edge in M has both ends in A_1 or

both ends in A_2. Let,

$$\mathcal{I} = \{(A_1, A_2) \in D(V) : \ (A_1, A_2) \ \text{ is covered by a matching}\}.$$

Then $D = (V, \mathcal{I})$, is a ditroid.

Example 2.2.3 $E = \{1,2,3\}$ and

$$\begin{aligned}\mathcal{I} = \{&(E,\phi),(1\,2,\phi),(2\,3,\phi),(3\,1,\phi),(1,\phi),\\ &(2,\phi),(3,\phi),(\phi,E),(\phi,1\,2),(\phi,2\,3),(\phi,3\,1),\\ &(\phi,1),(\phi,2),(\phi,3),(1,2),(2,3),(3,1),(\phi,\phi)\},\end{aligned}$$

then $D = (E, \mathcal{I})$ forms a ditroid.

The base family of the above ditroid is $\{(E,\phi),(\phi,E),\ (1,2),\ (2,3),\ (3,1)\}$. Thus unlike matroids, the cardinalities of the bases of a ditroid are not necessarily the same.

Qi, ([43], [44]) has shown that any ditroid $D = (E, \mathcal{I})$ gives rise to $2^{|E|}$ matroids. Let $D_S = (E, \mathcal{I}(S))$, for any $S \subseteq E$, where

$$\mathcal{I}(S) = \{(A_1 \cup A_2) \ : \ (A_1, A_2) \in \mathcal{I}, \ A \subseteq S, \ B \subseteq E/S\}. \tag{2.2.1}$$

$D = (E, \mathcal{I})$ is a ditroid if and only if D_S is a matroid, for any $S \subseteq E$. In particular, we have the following proposition from [44].

Proposition 2.2.1 If $D = (E, \mathcal{I})$ is a ditroid, then $M = (E, \mathcal{J})$ is a matroid, where

$$\mathcal{J} = \{X_1 \cup X_2 \ : \ (X_1, X_2) \in \mathcal{I}\}.$$

f an oracle exists for $M = (E, \mathcal{J})$, then it would be possible to find a base B of M such hat $|B| = \text{rank}(D)$, that is $|B| =$ cardinality of the maximum cardinality base of D.

In case a ditroid $D = (E, \mathcal{I})$ is constructed from two matroids $M_1 = (E, \mathcal{J}_1)$ and $M_2 = E, \mathcal{J}_2)$ as in example (2.2.1), then the union of matroids M_1 and M_2, that is $M_1 \vee M_2 = E, \mathcal{J})$ is the same matroid as obtained from $D = (E, \mathcal{I})$ by ignoring the signs of sets in $\mathcal{I}$, hat is $\mathcal{J} = \{X_1 \cup X_2 \ : \ (X_1, X_2) \in \mathcal{I}\}$.

In example (2.2.1), we saw that, if $M_1 = (E, \mathcal{J}_1)$ and $M_2 = (E, \mathcal{J}_2)$ are two matroids then $D = (E, \mathcal{I})$ is a ditroid, where :

$$\mathcal{I} = \{(A, B) \in D(E) \, : \, A \in \mathcal{J}_1, \, B \in \mathcal{J}_2)\}.$$

We will now show the converse relation, that is, any ditroid can be split into two matroids.

Let $D = (E, \mathcal{I})$ be a ditroid, then,

Theorem 2.2.1 Define $\mathcal{J}_1$ and $\mathcal{J}_2$ in the following way:

$$\mathcal{J}_1 = \{A \subseteq E \, : \, (A, B) \in \mathcal{I}\} \text{ and } \mathcal{J}_2 = \{B \subseteq E \, : \, (A, B) \in \mathcal{I}\},$$

then $M_1 = (E, \mathcal{J}_1)$ and $M_2 = (E, \mathcal{J}_2)$ are two matroids.

Proof. By (D1), we have $\phi \in \mathcal{I}$, that is $(\phi, \phi) \in \mathcal{I}$, implies $\phi \in \mathcal{J}_1$ and $\phi \in \mathcal{J}_2$.

Let, $Y_1 \in \mathcal{J}_1$ and $X_1 \subseteq Y_1$. We want to show that, $X_1 \in \mathcal{J}_1$.
Since $Y_1 \in \mathcal{J}_1$ there exists $Y_2 \in \mathcal{J}_2$, such that $(Y_1, Y_2) \in \mathcal{I}$.
Let $X = (X_1, Y_2)$. Since $X_1 \subseteq Y_1$ implies $X \subseteq Y$.
But $Y \in \mathcal{I}$ and this implies that $X \in \mathcal{I}$ and hence $X_1 \in \mathcal{J}_1$.
Thus, $\mathcal{J}_1$ satisfies the second axiom of matroids.

Let $X_1 \in \mathcal{J}_1$ and $Y_1 \in \mathcal{J}_1$ such that $|Y_1| = |X_1| + 1$. Clearly $X = (X_1, \phi)$and$Y = (Y_1, \phi) \in \mathcal{I}$. Now we have $|Y| = |X| + 1$, and also X, Y are non-cancelling, so by (D3), there exists $e_i \in Y/X$, such that $Z = (X + e_i) \in \mathcal{I}$. Clearly $Z = (X_1 + e_i, \phi)$. Thus $Z_1 = (X_1 + e_i) \in \mathcal{J}_1$. Hence $\mathcal{J}_1$ satisfies the third axiom of matroids. This completes the proof, that $M_1 = (E, \mathcal{J}_1)$ is a matroid. In a similar way, we can show that $M_2 = (E, \mathcal{J}_2)$ is also a matroid on E. □

Let $\text{rank}(M_1) = r_1$ and $\text{rank}(M_2) = r_2$. Then from construction of M_1 and M_2 it follows that $r_1 + r_2 \leq \text{rank}(M_1 \vee M_2)$, where $M_1 \vee M_2$ denotes the composition of two matroids M_1 and M_2. Let $D_1 = (E, \mathcal{I}_1)$ denote the ditroid, constructed from M_1 and M_2 as in example (2.2.1). Then

$$\text{rank}(D_1) = \text{rank}(M_1 \vee M_1) \geq r_1 + r_2.$$

2.2.1 When Will a System of Linear Inequalities Represent a Ditroid

Paralleling Conforti and Laurent's [8] result for matroids, we are able to prove a similar result for ditroids, i.e., given a system of linear inequalities $Ax \leq b$, where A is m×n matrix with $\{0, \pm 1\}$ entries and $b \in Z_+^m$, when will the $(0, \pm 1)$ vectors satisfying this set of inequalities represent a ditroid?

Theorem 2.2.2 Given a system of linear inequalities

$$Ax \leq b, \tag{$\mathcal{P}$}$$

where A is m×n matrix with $\{0, \pm 1\}$ entries and $b \in Z_+^m$. The rows of A are the characteristic vectors of disets in $D(E)$. The $(0, \pm 1)$ vectors in $\mathcal{P}$, define a ditroid if and only if

for all $i,\ j \in \{1,\ 2 \ldots, m\}$

$$b_i + b_j \geq r(A_{i.} \wedge A_{j.}) + r(A_{i.} \vee A_{j.}), \tag{2.2.2}$$

and for all j and u and v such that $A_{ju} = 1$, and $A_{jv} = -1$, $b_j \geq r(A_{j.} - e_u)$ and $b_j \geq r(A_{j.} + e_v)$.

Here, if $A_j.$ is the characteristic vector of (S, T), then $(A_j. - e_u)$ represents the characteristic vector of $(S/e_u, T)$, and $(A_j. + e_v)$ represents $(S, T/e_v)$, and

$$\begin{aligned} A_{i.} \wedge A_{j.} = (S_1, S_2) \quad &\text{where} \quad S_1 = \{u \ :\ A_{iu} = A_{ju} = 1\} \\ &\text{and} \quad S_2 = \{v \ :\ A_{iv} = A_{jv} = -1\}. \\ A_{i.} \vee A_{j.} = (T_1, T_2) \quad &\text{where} \quad T_1 = \{u \ :\ A_{iu} + A_{ju} \geq 1\} \\ &\text{and} \quad T_2 = \{v \ :\ A_{iv} + A_{jv} \leq -1\}. \end{aligned}$$

For any $(S, T) \in D(E)$, define

$$r(S, T) = \max\{x(S) - x(T) \ :\ x \in \mathcal{P}\}.$$

Proof. 2.2.2 holds for a matrix vector pair (A, b), only if it holds for each of the corresponding set of pairs $\{(A_i, \overline{b}_i)\}$ for $i = 1, 2 \ldots, n$, where A_i is the submatrix of A, obtained by deleting the i$^{\text{th}}$ column, and

$$\bar{b}_i = \begin{cases} b & \text{if there exists a solution to } \mathcal{P} \text{ with } x_i = 0. \\ b - A_{.i} & \text{if there exists a solution to } \mathcal{P} \text{ with } x_i = 1. \\ b + A_{.i} & \text{if there exists a solution to } \mathcal{P} \text{ with } x_i = -1. \end{cases}$$

We shall prove the theorem by induction on n.

Theorem is trivially true for $n \leq 2$. So, if the theorem is false, let p be the minimum value of n. Choose $\mathcal{P}$ for which $n = p$ and for which the theorem does not hold. Let $\mathcal{I}$ be the collection of disets corresponding to $\{0, \pm 1\}$ vectors in $\mathcal{P}$. We assume that $(E, \mathcal{I})$ is not a ditroid.

Since $b \geq 0$, $(\phi, \phi) \in \mathcal{I}$. Now if for some $(S,T) \in \mathcal{I}$, at least one subset of $(S,T) \notin \mathcal{I}$, then among all such sets of $D(E)$, take that (S,T) for which $|S \cup T|$ is minimum. Therefore, either $(S/u, T) \notin \mathcal{I}$, for some $u \in S$ or $(S, T/v) \notin \mathcal{I}$, for some $v \in T$.

In the first case, there exists i such that $b_i < r(A_{i.} - e_u)$, and in the second case, there exists j such that $b_j < r(A_{j.} + e_v)$, but in both cases, it is a contradiction. So if $(S,T) \in \mathcal{I}$ implies all subsets of (S,T) also belong to $\mathcal{I}$.

Hence the third axiom of ditroids must be violated, that is, there exist non-cancelling (S,T) and $(R,V) \in \mathcal{I}$, with $|(R,V)| = |(S,T)| + 1$, such that no augmentation is possible from (R,V) to (S,T). Then by minimality of p they are disjoint, and $|(S,T)\Delta(R,V)| = 3$.

Suppose $|(S,T)\Delta(R,V)| > 3$.
For $a \in (S,T)$ and $d \in (R,V)$, $(S,T)/a$ and $(R.V)/d$ also satisfy $\mathcal{P}$.
Since $\dim(E/\{a,d\}) < p$, $(S_1,T_1) = (S,T)/a$ and $(R_1,V_1) = (R,V)/d$ do not violate the third axiom for ditroids, that is, there exists $e \in (R_1,V_1)/(S_1,T_1)$ such that
$(S_1,T_1) + e = (S_2,T_2) \in \mathcal{I}$.
Since $|(S_2,T_2)| + 1 = |(R,V)|$, using the same arguments as above, there exists
$f \in (R,V)/(S_2,T_2)$ such that

$$(S_2,T_2) + f = (S_3,T_3) \in \mathcal{I}, \text{and } |(S_3,T_3)| = |(S,T)| + 1.$$

Also $|(S,T)\Delta(S_3,T_3)| = 3$. Again because of minimality of p, augmentation is possible from (S_3,T_3) to (S,T). But $(e,f) \in (S_3,T_3)/(S,T)$ and $e,f \in (R,V)$. This contradicts our

assumption that no augmentation is possible from (R,V) to (S,T). Thus $|(S,T)\Delta(R,V)| = 3$.

Following six cases are possible.

$$
\begin{array}{ll}
\text{(i)} & (S,T) = (i,\phi); \;\; (R,V) = (j\ k,\phi). \\
\text{(ii)} & (S,T) = (i,\phi); \;\; (R,V) = (j,k). \\
\text{(iii)} & (S,T) = (i,\phi); \;\; (R,V) = (\phi,j\ k,). \\
\text{(iv)} & (S,T) = (\phi,i); \;\; (R,V) = (j\ k,\phi). \\
\text{(v)} & (S,T) = (\phi,i); \;\; (R,V) = (j,k). \\
\text{(vi)} & (S,T) = (\phi,i); \;\; (R,V) = (\phi,j\ k,).
\end{array}
$$

case (i) $\quad (S,T) = (i,\phi),\ (R,V) = (j\ k,\phi)$.

Since no augmentation is possible from (R,V) to (S,T), there exist $A_{l.}$ and $A_{q.}$, tight at (S,T), such that

$$A_{li} = A_{lj} = 1 \quad \text{and} \quad b_l = 1.$$
$$A_{qi} = A_{qk} = 1 \quad \text{and} \quad b_q = 1.$$

So, $\quad b_l + b_q = 1 + 1 = 2,$

and, $r(A_{l.} \wedge A_{q.}) = 1, \qquad r(A_{l.} \vee A_{q.}) = 2$. This implies that,

$$b_l + b_q = 2 < r(A_{l.} \wedge A_{q.}) + r(A_{l.} \vee A_{q.}) = 3,$$

which is a contradiction.

case (ii) $\quad (S,T) = (i,\phi),\ (R,V) = (j,k)$.

Again since no augmentation is possible from (R,V) to (S,T), there exist $A_{s.}$ and $A_{r.}$, tight at (S,T), such that

$$A_{si} = A_{sj} = 1 \text{ and } b_s = 1.$$
$$A_{ri} = 1 \ \ A_{rk} = -1 \text{ and } b_r = 1.$$

So, $\quad b_s + b_r = 1 + 1 = 2,$

and, $r(A_{s.} \wedge A_{r.}) = 1, \ r(A_{s.} \vee A_{r.}) = 2$. Implies

$$b_s + b_r = 2 < r(A_{s.} \wedge A_{r.}) + r(A_{s.} \vee A_{r.}) = 3.$$

Again a contradiction.

case (iii) $(S,T) = (i,\phi),\ (R,V) = (\phi, j\,k)$.

No augmentation is possible from (R,V) to (S,T), so there exist $A_{\alpha.}$ and $A_{\beta.}$, tight at (S,T), such that

$$A_{\alpha i} = 1 \ \ A_{\alpha j} = -1 \ \text{ and } \ b_\alpha = 1.$$
$$A_{\beta i} = 1 \ \ A_{\beta k} = -1 \ \text{ and } \ b_\beta = 1.$$

So, $\quad b_\alpha + b_\beta = 1 + 1 = 2,$

and, $r(A_{\alpha.} \wedge A_{\beta.}) = 1,\ \ r(A_{\alpha.} \vee A_{\beta.}) = 2$. Implies

$$b_\alpha + b_\beta = 2 < r(A_{\alpha.} \wedge A_{\beta.}) + r(A_{\alpha.} \vee A_{\beta.}) = 3.$$

Which is a contradiction.

case (iv) $(S,T) = (\phi, i),\ (R,V) = (j\,k, \phi)$.

Again since no augmentation is possible from (R,V) to (S,T), there exist $A_{\gamma.}$ and $A_{\delta.}$, tight at (S,T), such that

$$A_{\gamma i} = -1 \ \ A_{\gamma j} = 1 \ \text{ and } \ b_\gamma = 1.$$
$$A_{\delta i} = -1 \ \ A_{\delta k} = -1 \ \text{ and } \ b_\delta = 1.$$

So, $\quad b_\gamma + b_\delta = 1 + 1 = 2,$

and, $r(A_{\gamma.} \wedge A_{\delta.}) = 1,\ \ r(A_{\gamma.} \vee A_{\delta.}) = 2$. Implies

$$b_\gamma + b_\delta = 2 < r(A_{\gamma.} \wedge A_{\delta.}) + r(A_{\gamma.} \vee A_{\delta.}) = 3.$$

Again a contradiction.

case (v) $(S,T) = (\phi, i),\ (R,V) = (j, k)$.

Again since no augmentation is possible from (R,V) to (S,T), there exist $A_{\lambda.}$ and $A_{\mu.}$, tight at (S,T), such that

$$A_{\lambda i} = 1 \ \ A_{\lambda j} = 1 \ \text{ and } \ b_\lambda = 1.$$
$$A_{\mu i} = -1 \ \ A_{\mu k} = -1 \ \text{ and } \ b_\mu = 1.$$

So, $\quad b_\lambda + b_\mu = 1 + 1 = 2,$

and, $r(A_{\lambda.} \wedge A_{\mu.}) = 1, \; r(A_{\lambda.} \vee A_{\mu.}) = 2$. Implies

$$b_\lambda + b_\mu = 2 < r(A_{\lambda.} \wedge A_{\mu.}) + r(A_{\lambda.} \vee A_{\mu.}) = 3.$$

Again a contradiction.

case (vi) $\quad (S,T) = (\phi, i), \; (R,V) = (\phi, j\,k)$.

Again since no augmentation is possible from (R,V) to (S,T), there exist $A_{\xi.}$ and $A_{\eta.}$, tight at (S,T), such that

$$A_{\xi i} = A_{\xi j} = -1 \text{ and } b_\xi = 1.$$
$$A_{\eta i} = A_{\eta k} = -1 \text{ and } b_\eta = 1.$$

So, $\quad b_\xi + b_\eta = 1 + 1 = 2,$

and, $r(A_{\xi.} \wedge A_{\eta.}) = 1, \; r(A_{\xi.} \vee A_{\eta.}) = 2$. Implies

$$b_\xi + b_\eta = 2 < r(A_{\xi.} \wedge A_{\eta.}) + r(A_{\xi.} \vee A_{\eta.}) = 3.$$

Again a contradiction.

In all the above cases, we reach at a contradiction, so the third axiom of ditroids must hold for $n = p$. Hence $(E, \mathcal{I})$ is a ditroid. $\square$

2.3 Augmentation Axioms and Bases

Qi, ([43], [44]), extended some of the matroid axioms to ditroids. We extend some more matroid axioms to ditroids and show the existence of an augmenting path between two independent sets of unequal cardinality.

A consequence of axiom (D3) is the following augmentation theorem for ditroids.

Theorem 2.3.1 Suppose that X and Y are independent in $D = (E, \mathcal{I})$ and are also non-cancelling and $|X| < |Y|$, then there exists $Z \subseteq Y/X$, such that $|X \cup Z| = |Y|$ and $(X \cup Z)$ is independent in $D = (E, \mathcal{I})$.

Proof. Let $|X \cup Z_0| = \max|X \cup Z|$ for all $Z \subseteq Y/X$ such that $X \cup Z \in \mathcal{I}$. Since X and Y are non-cancelling, $X \cup Z_0$ and Y are also non-cancelling. If $|X \cup Z_0| \geq |Y|$, select some $Z_0' \subseteq Z$ such that $|X \cup Z_0'| = |Y|$. But if $|X \cup Z_0| < |Y|$, then there exists $Y_0 \subseteq Y$ such that $|X \cup Z_0|+1 = |Y_0|$. Since $Y_0 \in \mathcal{I}$, by (D3) of ditroids there exists $e \in Y_0/(X \cup Z_0)$ such that $(X \cup Z_0) + e \in \mathcal{I}$. Since $e \notin X$, $Z_1 = Z_0 + e$ contradicts the maximality of Z_0. Hence the theorem. $\square$

Corollary 2.3.1 Let $Z = \{e_1, e_2, \ldots, e_p\}$; and X, Y be as in theorem (2.3.1). Then $X + e_i \in \mathcal{I}$ for all $i = 1, \ldots, p$. Here the signs of the elements in Z are attached to the elements themselves.

Qi, [44], proved the following lemma for an augmentation when the disets are cancelling.

Lemma 2.3.1 Suppose that $D = (E, \mathcal{I})$ is a ditroid, X and $Y \in \mathcal{I}$ with $|Y| = |X| + 1$. Then there exists $e_i \in Y/X$ and a $F = X \cap (-Y)$, such that we may add e_i from Y to X and invert F in X to form a new diset $Z \in \mathcal{I}$.

Proof Let $X = (A, B)$ and $Y = (C, D) \in \mathcal{I}$. There are two cases :

(i) $\overline{X} \subseteq \overline{Y}$. Since $|Y| = |X| + 1$. We have $e_i \in Y/X$ such that $\overline{Y} = \overline{X + e_i}$. Let $Z = Y$ and $F = X \cap (-Y)$. The conclusion holds.

(ii) $\overline{X} \not\subseteq \overline{Y}$. Let $X^1 = (A/D, B/C)$. Then X^1 and Y are non-cancelling and $|Y| = |X| + 1 \geq |X^1| + 1$. Thus we may form a new diset Y^1 by adding elements from Y to X^1, such that $|Y^1| = |Y|$ and $Y^1 \in \mathcal{I}$. Moreover, we have

$$|Y^1 \cap X| \geq |X^1| > |Y \cap X|, \tag{2.3.1}$$

and

$$\overline{Y^1}/\overline{X} \subseteq \overline{Y}/\overline{Y}. \tag{2.3.2}$$

If $\overline{X} \not\subseteq \overline{Y^1}$, we replace Y by Y^1 and continue this process. By (2.3.1), this process will terminate at a certain k, such that we have $Y^k \in \mathcal{I}$, $|Y^k| = |Y|$, $\overline{X} \subseteq \overline{Y^k}$ and

$$\overline{Y^k}/\overline{X} \subseteq \overline{Y}/\overline{X}.$$

Thus we have e_i such that $\overline{Y^k} = \overline{(X + e_i)}$ and $e_i \in Y/X$. Let $F = X \cap (-Y^k)$ and $Z = Y^k$. It is easy to see that

$$F \subseteq X \cap (-Y^{k-1}) \subseteq X \cap (-Y^{k-2}) \subseteq \cdots \subseteq X \cap (-Y).$$

Thus the conclusion also holds in this case.

We can interprect lemma 2.3.1 in terms of an **augmenting path** for X with respect to Y.

Let $P = (e_i, s_1, s_2, \ldots, s_k)$, where $e_i \in Y/X$, as in the lemma and $s_i \in F$, for all $i = 1, 2, \ldots, k$ and $|F| = k$.

We interprect $X \cup P$ as follows :

$$\begin{aligned} X + e_i &= (X_1 + e_i, X_2) \text{ or } (X_1, X_2 + e_i), \text{ according as } e_i \in Y_1 \text{ or} Y_2. \\ X + e_i + s_1 &= (X_1 + s_1, X_2/s_1) + e_i, \text{ in case } s_1 \in X_2 \\ &= (X_1/s_1, X_2 + s_1) + e_i, \text{ in case } s_1 \in X_1. \end{aligned}$$

The rest of $X \cup P$ can be constructed in a similar fashion.

Then $X \cup P = Z \in \mathcal{I}$ as has been shown in the lemma, and P is the augmenting path for X, since it alternately adds and subtracts elements to X. □

By repeated use of the above lemma, theorem 2.3.2, given below can be immediately proved.

Theorem 2.3.2 Let $X,\ Y \in \mathcal{I}$ be cancelling with $|X| < |Y|$ and $F \subseteq X.Y$. There exists $Z \subseteq Y/X$ such that $X^* \cup Z \in \mathcal{I}$ and $|X^* \cup Z| = |Y|$, where X^* is obtained from X by inverting F in X.

In example (2.2.3) we have seen that all the bases of a ditroid need not have the same cardinalities. But when two bases are non-cancelling, we have the following theorem.

Theorem 2.3.3 Any two non-cancelling bases of a ditroid have the same cardinality.

Proof. Let B_1 and B_2 be two bases of a ditroid $D = (E, \mathcal{I})$, where $B_1 = (B_{11}, B_{12})$ and

$B_2 = (B_{21}, B_{22})$, If possible let $|B_1| < |B_2|$.
We can always choose a set $B'_2 \subseteq B_2$ such that $|B_1| + 1 = |B'_2|$.
Clearly, B'_2 is an independent diset of $D = (E, \mathcal{I})$ and also non-cancelling with B_1, so by (D3), there exists $e \in B'_2/B_1$ such that $(B_1 + e) \in \mathcal{I}$.
Which contradicts the fact that B_1 is a base of $D = (E, \mathcal{I})$. Hence our assumption is wrong. This implies $|B_1| = |B_2|$. □

Problems of finding

1. a maximum cardinality base of a ditroid, and
2. a minimum cardinality base of a ditroid

were posed by Qi in [44], and have been intractable so far.

2.4 Rank Function of a Ditroid

Qi, [43], proved that the rank function h of a ditroid is bisubmodular and characterized it through the following rank theorem.

Ditroid rank theorem 2.4.1 An integral set-function $h : D(E) \longrightarrow \Re_+$ is a rank function of a ditroid, if and only if, it satisfies the following :

(i) $h(\phi, \phi) = 0$;
(ii) if $X \subseteq Y$, then $h(X) \leq h(Y)$;
(iii) $h(X) \leq |X|$;
(iv) h is a bisubmodular function on $D(E)$.

We obtain another equivalent characterization of a ditroid rank function, by extending the corresponding result for the matroid rank function.

Theorem 2.4.2 An integral set-function $h : D(E) \longrightarrow \Re_+$ is a rank function of a ditroid, if and only if, it satisfies the following :

(1) $h(\phi, \phi) = 0$;

(2) $h(X) \leq h(X + e_i) \leq h(X) + 1 \quad \forall\ e_i \notin X$.

(3) If $h(X) = h(X + e_i) = h(X + e_j)$ for $\forall\ e_i \neq e_j$ and e_i , $e_j \notin X$, then $h(X + e_i + e_j) = h(X)$, where e_i and e_j are signed elements.

Proof. Suppose h is the rank function of a ditroid D. (2) follows from (ii) and (iii) of theorem (2.4.1). And (3) follows from bisubmodularity of h.

Conversely, let h satisfy (1) ,(2) and (3). We will show that h is the rank function of some ditroid $D = (E, \mathcal{I}(h))$, where

$$\mathcal{I}(h) = \{X \in D(E) \text{ such that } h(X) = |X|\}.$$

Clearly, $\phi \in \mathcal{I}(h)$.

Let $A \in \mathcal{I}(h)$ and $B \subseteq A$. If possible $B \notin \mathcal{I}(h)$, then $h(B) < |B|$.

Therefore, if $\{e_1,\ e_2, \cdots, e_k\} = A/B$, we have

$$h(B + e_1) \leq h(B) + 1 < |B| + 1.$$

By repeated application of (2), we arrive at

$$h(A) = h(B \cup \{e_1,\ e_2, \cdots, e_k\}) < |B| + k = |A|.$$

Which is a contradiction. Hence, $\mathcal{I}(h)$ satisfies (D2).

Let $X, Y \in \mathcal{I}(h)$ be non-cancelling and $|X| + 1 = |Y|$, and

$$\begin{aligned}\overline{X} &= \{\overline{x_1}, \overline{x_2}, \cdots, \overline{x_q}, \overline{y_{q+1}}, \cdots, \overline{y_k}\} \\ \overline{Y} &= \{\overline{x_1}, \overline{x_2}, \cdots, \overline{x_q}, \overline{z_{q+1}}, \cdots, \overline{z_k}, \overline{z_{k+1}}\},\end{aligned}$$

where $y_i \neq z_j$, for any i and j.

Suppose that $(X + z_i) \notin \mathcal{I}(h)$; for any $q + 1 \leq i \leq k + 1$. Then, $h(X) = h(X \cup z_i) = |X|$, for all $i = q + 1$ to $k + 1$. Hence by (3)

$$h(X \cup \{z_i, z_j\}) = h(X) = |X|; \qquad \text{for } q + 1 \leq i,\ j \leq k + 1.$$

Applying (3) repeatedly, we get

$$h(Y) \leq h(X \cup \{z_{q+1}, \cdots, z_{k+1}\}) = |X| < |Y|.$$

This contradicts the fact that $Y \in \mathcal{I}(h)$. Hence $X + z_i \in \mathcal{I}(h)$, for some i and $\mathcal{I}(h)$ satisfies (D3). Thus $\mathcal{I}(h)$ is the collection of independent sets of a ditroid $D = (E,\mathcal{I}(h))$.

Let ρ be the rank function of the ditroid $D = (E,\mathcal{I}(h))$. To prove that h is the rank function of $D = (E,\mathcal{I}(h))$, it is enough to show

$$\rho(X) = h(X) \qquad \text{for all } \; X \in D(E).$$

By definition

$$\rho(X) = \max\{|A| \; : \; A \subseteq X, \; A \in \mathcal{I}(h)\} = |A^0| \text{ (say)}.$$

So A^0 is the maximal independent subset of X. If $X \notin \mathcal{I}(h)$, then $h(X) < |X|$. But A^0 is the maximal subset of X such that $A^0 \in \mathcal{I}(h)$. And by repeated use of (2) and (3) of this theorem, we get $h(X) = |A^0|$. Hence,

$$\rho(X) = h(X) \qquad \text{for all } \; X \in D(E). \; \square$$

Using the above two characterizations of the ditroid rank function h, it can be easily shown that h has the following properties.

Property 2.4.1 For $A \in D(E)$ and x, $y \notin A$

$$h(A \cup \{x,y\}) - h(A+x) \leq h(A+y) - h(A).$$

Property 2.4.2 For A, $B \in D(E)$ and $x \notin A$, and $x \notin B$,

$$h(A \cup B + x) - h(A \cup B) \leq h(A+x) - h(A).$$

Property 2.4.3 For A, B, $C \in D(E)$, and non-cancelling to each other,

$$h(A \cup B \cup C) - h(A \cup B) \leq h(A \cup C) - h(A).$$

The theorem given below is a generalisation of the corresponding result for matroids [53].

Theorem 2.4.3 Let $h : D(E) \to Z^+$ be a nondecreasing bisubmodular function with the property that $h(X) \leq |X|$ for all $X \in D(E)$, then

$$\mathcal{I}(h) = \{A \; : \; A \in D(E), \; h(A) = |A|\}$$

is the collection of independent disets of a ditroid on E. Moreover, h is the rank function of this ditroid.

Proof. In theorem (2.4.2), while proving its converse part, we had shown that the family

$$\mathcal{I}(h) = \{A : A \in D(E),\ h(A) = |A|\}$$

is the collection of independent disets of the ditroid $D = (E, \mathcal{I}(h))$, and we also proved that h is the rank function of the same ditroid. □

2.5 Circuits of A Ditroid

Definition 2.5.1 **Circuit** is a minimal dependent set of a ditroid $D = (E, \mathcal{I})$.

By definition, if we drop any element from a circuit, it will be an independent diset of ditroid $D = (E, \mathcal{I})$. We denote $\mathcal{C}$ as the collection of all circuits of a ditroid $D = (E, \mathcal{I})$. Here, we will give some examples of circuits of those ditroids which we discussed before.

Example 2.5.1 Circuits of the ditroid $D = (E, \mathcal{I})$, given in example (2.2.1), are the following :

$$\mathcal{C} = \begin{cases} (C_1, \phi) & \text{when } C_1 \text{ is a circuit of } M_1 \\ (\phi, C_2) & \text{when } C_2 \text{ is a circuit of } M_2. \end{cases}$$

Example 2.5.2 In example (2.2.2), the circuits of D are the collection of those diset of $D(V)$, which are not covered by a matching but will be covered by a matching when we drop any one node from those disets.

Example 2.5.3 The disets $\{(2,1); (3,2); (1,3)\}$ are the circuits of example (2.2.3).

From the above examples, we have three types of circuit, first of the type $C = (C_1, \phi)$, where $C_1 \neq \phi$, second of the type $C = (\phi, C_2)$, where $C_2 \neq \phi$ and third of the type $C = (C_1, C_2)$, where $C_1 \neq \phi$ and $C_2 \neq \phi$. And in each case C_1, C_2 are the subsets of the ground set E.

It is possible to characterize a ditroid with respect to its circuit axioms. It is obvious

that every dependent diset of a ditroid $D = (E, \mathcal{I})$ must contain a circuit. This and some other easy observations are listed in the following theorem.

Theorem 2.5.1 The following statements are true for a circuit of a ditroid $D = (E, \mathcal{I})$ with h as its rank function.

(i) If C is a circuit, then $h(C) = |C| - 1$.

(ii) If C is a circuit, then there exists a base B of the ditroid, such that $|C| \leq h(B) + 1$.

(iii) Every proper subset of a circuit is an independent diset of the ditroid.

Proof. (i) Follows from the definition of circuits.

(ii) If B is a base of D, then $B + e$ is a dependent set for any $e \in E/\overline{B}$. Thus $B + e$ must contain a circuit.

(iii) Follows from the definition of circuits. □

Theorem 2.5.2 (Circuit axioms of a ditroid.)

(C1) If C_1 and C_2 are two distinct circuits of a ditroid $D = (E, \mathcal{I})$ then $C_1 \not\subseteq C_2$ and $C_2 \not\subseteq C_1$.

(C2) If $C_1 = (C_{11}, C_{12})$ and $C_2 = (C_{21}, C_{22})$ are two circuits of D such that $C_{11} \cap C_{22} = C_{12} \cap C_{21} = \phi$, that is C_1 and C_2 are non-cancelling and $z \in C_1 \cap C_2$, then there exists a circuit C_3 such that $C_3 \subseteq (C_1 \cup C_2)/z$.

(C3) If $C_1 = (C_{11}, C_{12})$ and $C_2 = (C_{21}, C_{22})$ are two non-cancelling circuits of D and $y \in C_1/C_2$ then for each $x \in C_1 \cap C_2$ there exists a circuit C_3 such that $y \in C_3 \subseteq (C_1 \cup C_2)/x$.

Proof of (C1). If possible let $C_1 \subseteq C_2$. Then there exists at least one element $e \in C_2/C_1$ such that $C_1 \subseteq C_2/e$ which contradicts the fact that C_2 is a circuit. Similarly, $C_2 \not\subseteq C_1$.

Proof of (C2). It is necessary that C_1 and C_2 be non-cancelling. Otherwise, $C_1 \cup C_2$ may be independent set. Consider the circuits of the ditroid D given in examples (2.5.3). Let $C_1 = (2, 1)$; $C_2 = (3, 2)$, then $C_1 \cup C_2 = (3, 1) \in \mathcal{I}$.

Suppose there is no C_3 such that $C_3 \subseteq (C_1 \cup C_2)/z$, for all $z \in C_1 \cap C_2$. This implies that for

all $z \in C_1 \cap C_2$, $(C_1 \cup C_2)/z \in \mathcal{I}$, that is $h((C_1 \cup C_2)/z) = |C_1 \cup C_2| - 1$. But $h(C_1) = |C_1| - 1$ and $h(C_2) = |C_2| - 1$, and bisubmodularity of h gives

$$\begin{aligned} h(C_1 \cup C_2) + h(C_1 \cap C_2) &\leq h(C_1) + h(C_2) = |C_1| + |C_2| - 2 \\ &= |C_1 \cup C_2| + |C_1 \cap C_2| - 2. \qquad [C_1 \& C_2 \text{ non-cancelling}] \end{aligned}$$

But we know that $h(C_1 \cup C_2) \geq h((C_1 \cup C_2)/z) = |(C_1 \cup C_2)| - 1$. This implies that $h(C_1 \cap C_2) < |(C_1 \cap C_2)| - 1$. Which is a contradiction, since $(C_1 \cap C_2)$ is an independent diset. Hence the proof of (C2).

Proof of (C3). Suppose that C_1, C_2, x, y are such that the above statement (in (C3)) is not true and that $|C_1 \cup C_2|$ is minimal with this property. By (C2) there exists a circuit C_3 such that $C_3 \subseteq (C_1 \cup C_2)/x$, but for all $y \in C_1/C_2$, $y \notin C_3$. Now $C_3 \cap (C_2/C_1)$ can not be null, otherwise $C_3 \subseteq C_1$.

Let $z \in C_3 \cap (C_2/C_1)$. Consider $C_2, C_3, z \in C_3 \cap C_2, x \in C_2/C_3$ and since $y \notin C_2 \cap C_3$, so $C_2 \cup C_3$ is a proper subset of $C_1 \cup C_2$, therefore by minimality of $|C_1 \cup C_2|$, there exists a circuit C_4 such that $x \in C_4 \subseteq (C_2 \cup C_3)/z$.

Now consider $C_1, C_4 . x \in C_1 \cap C_4, y \notin C_2 \cup C_3$, and hence $y \in C_1/C_4$. Also $C_1 \cup C_4 \subseteq C_1 \cup C_3$. Therefore by the minimality argument again there exists a circuit C_5 such that $y \in C_5 \subseteq (C_1 \cup C_4)/x$. Which is a contradiction. □

Lemma 2.5.1 If A is independent in D, then for any $e \in E$; $A + e$ contains at most one circuit.

Proof. Either $A + e$ is independent or dependent. If it is independent, it contains no circuit. When $A + e$ is dependent, if possible let two circuits $C_1 = (C_{11}, C_{12})$ and $C_2 = (C_{21}, C_{22})$ be contained in $A + e$, that is $C_1 \subseteq A + e$ and $C_2 \subseteq A + e$. This implies that $C_1 \cup C_2 \subseteq A + e$ and $e \in C_1 \cap C_2$.

Since C_1 and C_2 are non-cancelling and $e \in C_1 \cap C_2$, by circuit axioms (C2), there exists $C_3 \subseteq (C_1 \cup C_2)/e$. But $C_1 \cup C_2/e \subseteq A$.

This contradicts the fact that A is an independent diset of D. Hence the lemma. □

It is now possible to characterize a ditroid with respect to its circuits.

Theorem 2.5.3 A collection $\mathcal{C}$ of disets of E is the set of circuits of a ditroid on E, if

and only if the disets in $\mathcal{C}$ satisfy axioms (C1) and (C2).

Proof We define a function h associated with $\mathcal{C}$ and then we will show that h is the rank function of a ditroid.

For any $A \in D(E)$, where $A = \{x_1, x_2, \ldots, x_r\}$ and the elements of A are ordered. Set $\theta_i = 0$. If $\{x_1, x_2, \ldots, x_i\}$ contains a member $C \in \mathcal{C}$, such that $x_i \in C$ and set $\theta_i = 1$, otherwise. We define

$$h(A) = \sum_{i=1}^{r} \theta_i.$$

For any permutation π of $(1, 2, \ldots, r)$

$$h(x_1, x_2, \ldots, x_r) = h(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(r)}).$$

Above claim will be true, if we can show that

$$h(x_1, x_2, \ldots, x_{r-2}, x_{r-1}, x_r) = h(x_1, x_2, \ldots, x_{r-2}, x_r, x_{r-1}).$$

Let Y be any diset of $\{x_1, x_2, \ldots, x_{r-2}\}$. Assume

$h(Y) = a$, $h(Y + x_{r-1}) = a_1$, $h(Y + x_r) = a_2$, $h(Y + x_{r-1} + x_r) = a_{12}$, $h(Y + x_r + x_{r-1}) = a_{21}$.

Signs of x_{r-2}, x_{r-1} and x_r will remain same in the above three disets, whatever they are.

case 1 Let there be no member of $\mathcal{C}$ in $(Y + x_{r-1})$ containing x_{r-1} and none in $(Y + x_r)$ containing x_r then

$$a_1 = a_2 = a + 1.$$

If there is a member of $\mathcal{C}$ in $(Y + x_{r-1} + x_r)$, containing x_{r-1} and x_r then,

$$a_{12} = a_1 = a_2 = a_{21},$$

othersise

$$a_{12} = a_1 + 1 = a_2 + 1 = a_{21}.$$

case 2 There is a member of $\mathcal{C}$ say C_2 contained in $(Y + x_{r-1})$ and $x_{r-1} \in C_2$ and there is a member C_1 of $\mathcal{C}$ in $(Y + x_{r-1} + x_r)$ containing x_r and x_{r-1}, then by (C3), there exists $C_3 \in \mathcal{C}$ such that $x_r \in C_3 \subseteq (Y + x_r)$. Hence

$$a_{12} = a_1 = a = a_2 = a_{21}.$$

case 3 There is a C_2 as in case-2, but no C_1 as in case-2., if C_3 exists as above then,

$$a_{12} = a_1 = a = a_2 = a_{21},$$

otherwise

$$a_{12} = a_1 + 1 = a + 1 = a_2 = a_{21}.$$

case 4 There is a member $C \in \mathcal{C}$ such that $x_r \in C \subseteq Y + x_r$, and there is a member C_1 of $\mathcal{C}$ in $(Y + x_{r-1} + x_r)$ contaning x_r and x_{r-1}. This is the same as case-2. And in case there is no C_1 of $\mathcal{C}$ in $(Y + x_{r-1} + x_r)$ contaning x_r and x_{r-1}, then this case reduces to case-3.

Finally, we get

$$h(x_1, x_2, \ldots, x_r) = h(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(r)}),$$

so h is well defined.

Let

$$\mathcal{I} = \{X \in D(E) \, : \, C \not\subseteq X \, \forall C \in \mathcal{C}\}.$$

By definition $\mathcal{I}$ and h, $h(Y) \leq h(X)$, for all $Y \subseteq X \in \mathcal{I}$. Let $A \in D(E)$.
Suppose $h(A + x) = h(A + y) = h(A)$, for some signed elements x and y, then there exists C_1 and C_2 of $\mathcal{C}$ such that

$$x \in C_1 \subseteq A + x, \text{ and } y \in C_2 \subseteq A + y.$$

By definition of h, it follows that

$$h(A + x + y) = h(A).$$

Thus by theorem (2.4.2) it follows that $D = (E, \mathcal{I})$ is a ditroid. □

2.6 Different Operations on Ditroids

In this section we define various operations, that is deletion, contraction, reflection and restriction (or projection) on a ditroid, and will show that the new diset systems, again define ditroids. Let $D = (E, \mathcal{I})$ be a ditroid on E, and h be its rank function.

2.6.1 Deletion

Let $S \subseteq E$ and $E^S = E/S$. Define $\mathcal{I}^S$ in the following way:

$$\mathcal{I}^S = \begin{cases} A & : A \in \mathcal{I} \text{ and } \overline{A} \cap S = \phi \\ A' & : \exists\ A \in \mathcal{I} \text{ such that } A' \subseteq A \text{ and } \overline{A} \cap S \neq \phi \text{ but } \overline{A'} \cap S = \phi. \end{cases}$$

This is called deletion of D with respect to S.

Now we will show that $D^S = (E^S, \mathcal{I}^S)$ is a ditroid and is known as the deletion ditroid of D with respect to S.

Proof. $\phi \in \mathcal{I}^S$.
Let $X' \in \mathcal{I}^S$ and $Y' \subseteq X'$. We will show that $Y' \in \mathcal{I}^S$.
Since $\overline{X'} \cap S = \phi$ implies $\overline{Y'} \cap S = \phi$. This shows that $Y' \in \mathcal{I}^S$.
Hence $\mathcal{I}^S$ satisfies (D2).
Let X' and $Y' \in \mathcal{I}^S$ be non-cancelling with $|X'| = |Y'| + 1$.
Since, X' and $Y' \in \mathcal{I}^S$, implies X' and $Y' \in \mathcal{I}$. By (D3) there exists $e \in X'/Y'$ such that $(Y' + e) \in \mathcal{I}$, and $(Y' + e) \in \mathcal{I}^S$. Hence $\mathcal{I}^S$ satisfies (D3). This shows that $D^S = (E^S, \mathcal{I}^S)$ is a ditroid. □

Let us define h^S as follows :

$$h^S(A) = \begin{cases} h(A) & \text{if } A \in \mathcal{I} \text{ and } \overline{A} \cap S = \phi \\ h(A/(S,\phi)) & \text{if } A \in \mathcal{I} \text{ and } \overline{A} \cap S \neq \phi. \end{cases}$$

In order to show that $h^S(A)$ is the rank function for the deletion ditroid D^S, we need only to verify that h^S is bisubmodular on $D(E^S)$.
For $A, B \in D(E^S)$,

$$\begin{aligned} h^S(A) + h^S(B) &= h(A/(S,\phi) + h(B/(S,\phi) \\ &\geq h(A_1 \cup B_1/S \cup A_2 \cup B_2), A_2 \cup B_2/S \cup A_1 \cup B_1)) \\ &\qquad + h(A_1 \cap B_1/S, A_2 \cap B_2/S) \\ &= h(A \cup B/(S,\phi)) + h(A \cap B/(S,\phi)) \end{aligned}$$

$$= h^S(A \cup B) + h^S(A \cap B).$$

This shows that h^S is bisubmodular. □

Oracle for the deletion ditroid can also be constructed from an oracle for D.

2.6.2 Contraction

Let $S \subseteq E$, define $\mathcal{I}_S$ as follows :

$$\mathcal{I}_S = \{Y \in D(S) \ : \ \exists \text{ a maximal diset } X \in D(E/S) \text{ such that } X \cup Y \in \mathcal{I}\}.$$

This is called contraction of D to S. Next we will show that $D_S = (E, \mathcal{I}_S)$ is a ditroid, and is known as contraction ditroid of $D = (E, \mathcal{I})$.

Proof. $\phi \in \mathcal{I}_S$.

Let $Y \subseteq X$ and $X \in \mathcal{I}_S$, we will show that $Y \in \mathcal{I}_S$. Since $X \in \mathcal{I}_S$, by definition there exists a diset $A \in D(E/S)$, which is maximal, such that $X \cup A \in \mathcal{I}$.

We have $Y \subseteq X \ \Rightarrow \ Y \subseteq (X \cup A)$.

Either A is a maximal diset belonging to $D(E/S)$, for which $Y \cup A \in \mathcal{I}$ or A is not maximal. In case A is not maximal it is always possible to augment A to $A' \in D(E/S)$ which is a maximal diset in $D(E/S)$ such that $Y \cup A' \in \mathcal{I}$. Hence $Y \in \mathcal{I}_S$.

Let $X,\ Y \in \mathcal{I}_S$ be non-cancelling with $|X| = |Y| + 1$. So there exist two maximal disets $A,\ B \in D(E/S)$ such that

$$X \cup A \in \mathcal{I} \qquad \text{and} \qquad Y \cup B \in \mathcal{I}.$$

This gives $X,\ Y \in \mathcal{I}$ and by (D3), we can augment Y to $Y + e$, such that $(Y + e) \in \mathcal{I}$ and $e \in X/Y$.

Now, either B will be a maximal diset belonging to $D(E/S)$, for which $(Y + e) \cup B \in \mathcal{I}$ or $(Y + e) \cup B \notin \mathcal{I}$. In the latter case, there exists a circuit $C \subseteq (B + e)$. Drop some $e' \in C$. Then $(Y + e) \cup (B/e') \in \mathcal{I}$. If B/e' is a maximal diset belonging to $D(E/S)$ we are done, if not, we can always augment B/e' to some $B' \in D(E/S)$ such that $(Y + e) \cup B' \in \mathcal{I}$, and B' is a maximal diset belonging to $D(E/S)$. Hence $Y + e \in \mathcal{I}_S$. So $\mathcal{I}_S$ satisfies (D3).

This proves that $D = (E, \mathcal{I}_S)$ is a ditroid. □

Let,

$$h_S(Y) = \max_{X \subseteq Y} \{|X| \ : \ X \in \mathcal{I}_S\}.$$

From the definition of contraction we have, $h_S(Y) = h(Y)$. Hence h_S is the rank function of the contraction ditroid $D_S = (E, \mathcal{I}_S)$.

Oracle for the contraction ditroid can also be constructed from an oracle for D.

2.6.3 Reflection

Let $S \subseteq E$. For each $X \in D(E)$, let X' be the diset of E, obtained by reflecting X with respect to the elements belonging to S, that is $X'(e) = X(e)$ if $e \notin S$ and $X'(e) = -X(e)$ if $e \in S$.

This is known as reflection of D with respect to S.

Now we will show that $D' = (E, \mathcal{I}')$, that is reflection of D with respect to S, is also a ditroid, where

$$\mathcal{I}' = \{X' \ : \ X \in \mathcal{I}\}.$$

Proof. $\phi \in \mathcal{I}'$.

Let $X' \in \mathcal{I}'$ and $Y' \subseteq X'$, we will show that $Y' \in \mathcal{I}'$.

Since, $X' \in \mathcal{I}'$, there exists $X \in \mathcal{I}$, such that X' is its reflection. We can choose a subset $Y \subseteq X$ such that Y' is its reflection. This implies $Y' \in \mathcal{I}'$. Hence, $\mathcal{I}'$ satisfies (D2).

Suppose Y', $X' \in \mathcal{I}'$ with $|X'| = |Y'| + 1$ and are non-cancelling. Then there exist $X, Y \in \mathcal{I}$ with $|X| = |Y| + 1$ and X and Y are non-cancelling. Using (D3), we find a new diset $Z \in \mathcal{I}$, where $Z = Y + e$ and $e \in X/Y$. So $e \in X'/Y'$.

Now, $Y' + e$ is the reflection of $Y + e$, since, if $e \in S$, then sign of e is changed in $Y + e$, and if $e \notin S$ sign of e does not change. And $Y + e \in \mathcal{I}$. This shows that $Y' + e \in \mathcal{I}$. Hence $D' = (E, \mathcal{I}')$ is a ditroid. □

If we define h' as :

$$h'(A') = h(A),$$

where A' is the reflection of A with respect to S, then it can be easily verified that h' is the

rank function of $D' = (E, \mathcal{I}')$.

2.6.4 Restriction or Projection

Let $T \in D(E)$, define $\mathcal{I}_T = \{ A \cap T \ : \ A \in \mathcal{I} \}$.
This is called the restriction (or projection) of D on T.
We will prove that $D_T = (\widetilde{T}, \mathcal{I}_T)$, which is the restriction (or projection) of D on T, is a ditroid.

Proof. It is clear that $\mathcal{I}_T$ is a subfamily of $\mathcal{I}$ and it can be easily verified that it will satisfy the three ditroid axioms. And $h_T(A) = h(A \cap T)$, will be its corresponding rank function. □

2.7 Some Special Kinds of Ditroids

2.7.1 Composition of Ditroids

In this subsection we will discuss two types of compositions of ditroids.

(1) Let $D_0 = (E, \mathcal{I}_0)$ and $D_1 = (E, \mathcal{I}_1)$ be two ditroids, define $\mathcal{I}$ as follows:

$$\mathcal{I} = \{A = A_0 \cup A_1 \ : \ A_0 \in \mathcal{I}_0, A_1 \in \mathcal{I}_1 \text{ and } A_0 \cap A_1 = A_0 \cap (-A_1) = \phi\}. \qquad (2.7.1)$$

Theorem 2.7.1 $D = (E, \mathcal{I})$ is a ditroid, i.e., composition of two ditroids is again a ditroid.

Proof. Clearly $\phi \in \mathcal{I}$.
Let $Y \subseteq X$ for $X \in \mathcal{I}$. We will show that $Y \in \mathcal{I}$.
Since $X \in \mathcal{I}$ implies there exists $X_0 \in \mathcal{I}_0$ and $X_1 \in \mathcal{I}_1$ such that $X = X_0 \cup X_1$ and $X_0 \cap X_1 = X_0 \cap (-X_1) = \phi$. Choose Y_0 and Y_1 in the following way:

$$Y_0 = Y \cap X_0 \text{ and } Y_1 = Y \cap X_1.$$

Since $\mathcal{I}_0$ and $\mathcal{I}_1$ are the collection of independent disets of ditroids, $Y_0 \in \mathcal{I}_0$, $Y_1 \in \mathcal{I}_1$ and $Y_0 \cap Y_1 = Y_0 \cap (-Y_1) = \phi$, and $Y = Y_0 \cup Y_1$. Thus $Y \in \mathcal{I}$.
To prove (D3), let X, $Y \in \mathcal{I}$ be non-cancelling with $|X| = |Y| + 1$.

X and $Y \in \mathcal{I}$, implies there exists X_0, X_1, Y_0, Y_1 such that X_0, $Y_0 \in \mathcal{I}_0$ and X_1, $Y_1 \in \mathcal{I}_1$ and $X_0 \cap X_1 = X_0 \cap (-X_1) = \phi$ and $Y_0 \cap Y_1 = Y_0 \cap (-Y_1) = \phi$ and $X = X_0 \cup X_1$, $Y = Y_0 \cup Y_1$.
X and Y non-cancelling implies X_0 and Y_0, and X_1 and Y_1 are non-cancelling.
Since $|X| = |Y| + 1$, either $|X_0| > |Y_0|$ or $|X_1| > |Y_1|$.
For definiteness, let $|X_0| > |Y_0|$.
From corollary (2.3.1) there exists at least one $e_1 \in X_0/Y_0$ such that $Y_0 + e_1 \in \mathcal{I}_0$.
We have two cases. Either (1) $e_1 \notin Y_1$ or (2) $e_1 \in Y_1$.

case (1) $\quad e_1 \notin Y_1$,
then $Y_0^1 = Y_0 + e_1 \in \mathcal{I}_0$ and $Y_1 \in \mathcal{I}_1$
also $Y_0^1 \cap Y_1 = Y_0^1 \cap (-Y_1) = \phi$.
Therefore $Y^1 = Y + e_1 \in \mathcal{I}$. Hence $\mathcal{I}$ satisfies (D3).

case (2) $\quad e_1 \in Y_1$.
Let $Y_0^1 = Y_0 + e_1 \in \mathcal{I}_0$, then $Y_0^1 \cap Y_1 \neq \phi$, and $Y = (Y_0 + e_1) \cup Y_1 = (Y_0 \cup Y_1)$ i.e., no augmentation has taken place.

Now, drop e_1 from Y_1 that is let $Y_1^1 = Y_1 - e_1 \in \mathcal{I}_0$ and let $Y_0^1 = Y_0 + e_1 \in \mathcal{I}_0$.
If $|X_0| > |Y_0^1|$, then there exists an $e_2 \in X_0/Y_0^1$, such that
$Y_0^2 = Y_0^1 + e_2 \in \mathcal{I}_0$
If $e_2 \notin Y_1^1$, we are done. But if $e_2 \in Y_1^1$, then drop e_2 from Y_1^1, let $Y_1^2 = Y_1^1 - e_2$ and $Y_0^2 = Y_0^1 + e_2$.
This process may get repeated at most $|X_0| - |Y_0| = r$ times, without any augmentation. Let $Y = (Y_0^r \cup Y_1^r)$, at the end of this process. Since $|X_0| = |Y_0^r|$, this implies that $|X_1| = |Y_1^r| + 1$.
By axiom (D3), there exists $e_{r+1} \in X_1/Y_1^r$ such that $Y_1^r + e_{r+1} \in \mathcal{I}_1$.
Two sub cases are possible. Either (i) $e_{r+1} \notin Y_0^r$, or (ii) $e_{r+1} \in Y_0^r$.

sub case (i) $\quad e_{r+1} \notin Y_0^r$.
In this case $Z = (Y_0^r \cup Y_1^r + e_{r+1}) \in \mathcal{I}$ is the required augmenting diset of Y.

Hence, $\mathcal{I}$ satisfies (D3).

sub case (ii) $\quad e_{r+1} \in Y_0^r$.

Here $Y_0^r \cap (Y_1^r + e_{r+1}) \neq \phi$, and $Y = Y_0^r \cup (Y_1^r + e_{r+1})$ i.e., no augmentation takes place.

Let $\{x_1, x_2, \cdots, x_p\} = X_0/Y_0^r$ and $\{y_1, y_2, \cdots, y_q\} = X_1/Y_1^r$.

Since, $|X_1| = |Y_1^r| + 1$ there exists $e_{r+1} \in X_1/Y_1^r$ such that $Y_1^r + e_{r+1} \in \mathcal{I}_1$.

And e_{r+1} is one of y_j's for $j = 1$ to q.

For convenience, let $e_{r+1} = y_1$.

Add e_{r+1} to Y_1^r, and drop e_{r+1} from Y_0^r, i.e., $Y_1^{r+1} = Y_1^r + e_{r+1}$, and $Y_0^{r+1} = Y_0^r - e_{r+1}$.

Now $|X_1| = |Y_1^{r+1}|$, and $|X_0| = |Y_0^{r+1}| + 1$,

and the set, $X_0/Y_0^{r+1} = X_0/Y_0^r$ but $X_1/Y_1^{r+1} = \{y_2, y_3, \cdots, y_q\}$.

Since $|X_0| = |Y_0^{r+1}| + 1$, there exists $e_{r+2} \in X_0/Y_0^{r+1}$ such that $Y_0^{r+1} + e_{r+2} \in \mathcal{I}_0$.

If $e_{r+2} \notin Y_1^{r+1}$, we are done, otherwise we add e_{r+2} to Y_0^{r+1} and drop e_{r+2} from Y_1^{r+1}, that is $Y_0^{r+2} = Y_0^{r+1} + e_{r+2}$ and $Y_1^{r+2} = Y_1^{r+1} - e_{r+2}$.

Clearly, e_{r+2} must be one of x_i's for $i = 1$ to p.

Let $e_{r+2} = x_1$ and the set, $X_1/Y_1^{r+2} = X_1/Y_1^{r+1}$ but $X_0/Y_0^{r+2} = \{x_2, x_3, \cdots, x_p\}$

and $|X_0| = |Y_0^{r+2}| \Rightarrow |X_1| = |Y_1^{r+2}| + 1$.

We alternately keep adding and subtracting elements from Y_0^r and Y_1^r, till an augmentation takes place either in Y_0^r or Y_1^r. Hence $\mathcal{I}$ satisfies (D3). □

We define the rank function of D, in terms of the rank functions of D_0 and D_1. Let h_0, h_1 and h be the rank functions of D_0, D_1 and D respectively, then

$$h(X) = \min_{Y \subseteq X} \{ h_0(Y) + h_1(Y) + |X/Y| \}.$$

We will show that h satisfies the ditroid rank theorem.

Proof.

(i) $\quad h(\phi) = \min_{Y \subseteq \phi} \{ h_0(Y) + h_1(Y) + |\phi/Y| \}$.

But $\phi \subseteq \phi \quad \Rightarrow \quad h(\phi) = h_0(\phi) + h_1(\phi) + |\phi| = 0$.

(ii) $\quad$ Let $Y \subseteq X$, then

$$h(Y) = \min_{B \subseteq Y} \{ h_0(B) + h_1(B) + |Y/B| \}$$

$$= \quad h_0(B^\star) + h_1(B^\star) + |Y/B^\star| \quad \text{for some } B^\star \subseteq Y.$$

And

$$\begin{aligned}
h(X) &= \min_{A \subseteq X} \{\, h_0(A) + h_1(A) + |X/A| \,\} \\
&= h_0(A^\star) + h_1(A^\star) + |X/A^\star| \quad \text{for some } A^\star \subseteq X \\
&\geq h_0(A^\star) + h_1(A^\star) + |Y/A^\star|, \qquad\qquad (2.7.2) \\
&\qquad \text{since } |X/A^\star| \geq |Y/A^\star|.
\end{aligned}$$

In case $A^\star \subseteq Y$, by definition of h

$$h(Y) < h_0(A^\star) + h_1(A^\star) + |Y/A^\star|,$$

and if $Y \subseteq A^\star$, then $B^\star \subseteq A^\star$ and from monotonicity of h_0 and h_1, we get

$$h_0(A^\star) + h_1(A^\star) + |Y/A^\star| \geq h_0(B^\star) + h_1(B^\star) + |Y/B^\star|.$$

Now from (2.7.2) we get

$$h(X) \geq h_0(B^\star) + h_1(B^\star) + |Y/B^\star| = h(Y).$$

Hence if $Y \subseteq X \;\Rightarrow\; h(Y) \leq h(X)$.

(iii) Let X and Y be any two disets of E. Then,

$$\begin{aligned}
h(X) + h(Y) &= \min_{A \subseteq X,\, B \subseteq Y} \{\, h_0(A) + h_1(A) + h_0(B) + h_1(B) + |X/A| + |Y/B| \,\} \\
&= h_0(A^\star) + h_1(A^\star) + h_0(B^\star) + h_1(B^\star) + |X/A^\star| + |Y/B^\star| \\
&\qquad\qquad \text{for some } A^\star \subseteq X \text{ and } B^\star \subseteq Y \\
&\geq h_0(A^\star \cup B^\star) + h_0(A^\star \cap B^\star) + h_1(A^\star \cup B^\star) + h_1(A^\star \cap B^\star) + \\
&\qquad |(X/A^\star) \cup (Y/B^\star)| + |(X/A^\star) \cap (Y/B^\star)| \\
&= h_0(A^\star \cup B^\star) + h_0(A^\star \cap B^\star) + h_1(A^\star \cup B^\star) + h_1(A^\star \cap B^\star) + \\
&\qquad |(X \cup Y)/(A^\star \cup B^\star)| + |(X \cap Y)/(A^\star \cap B^\star)| \\
&\geq h(X \cup Y) + h(X \cap Y),
\end{aligned}$$

since $A^\star \cup B^\star \subseteq X \cup Y$ and $A^\star \cap B^\star \subseteq X \cap Y$. Hence h is bisubmodular.

(iv) Since $h(X) = \min_{Y \subseteq X} \{\, h_0(Y) + h_1(Y) + |X/Y| \,\}$,

Putting $Y = \phi$, gives $\quad h(X) \le h_0(\phi) + h_1(\phi) + |X|$, i.e. $h(X) \le |X|$.

This shows that h satisfies the ditroid rank theorem. $\square$

(2) Let $D_0 = (E_0, \mathcal{I}_0)$ and $D_1 = (E_1, \mathcal{I}_1)$, be two ditroids, and let $E = E_0 \Delta E_1$, and $S = E_0 \cap E_1$. Define $\mathcal{I}$ as follows:

$$\mathcal{I} = \{A_0 \Delta A_1 \;:\; A_0 \in \mathcal{I}_0, A_1 \in \mathcal{I}_1, \text{ and } A_0|_S = A_1|_S, \text{ i.e., } A_0 \text{ and } A_1 \text{ agree on } S\}. \quad (2.7.3)$$

Theorem 2.7.2 $\quad D = (E, \mathcal{I})$, where $\mathcal{I}$ as given in (2.7.3), is a ditroid.

Proof. Let ϕ_0 and ϕ_1 denote the null disets of D_0 and D_1 respectively, and $\phi_0|_S = \phi_1|_S$, so $\phi_0 \Delta \phi_1 = \phi \in \mathcal{I}$. Hence $\mathcal{I}$ satisfies (D1).

Let $Y \subseteq X$ and $X \in \mathcal{I}$. We will show that Y also belongs to $\mathcal{I}$.

Since $X \in \mathcal{I}$ implies, there exists $X_0 \in \mathcal{I}_0$ and $X_1 \in \mathcal{I}_1$,

such that $X_0|_S = X_1|_S$ and $X = X_0 \Delta X_1$.

$Y \subseteq X \;\Rightarrow\; Y \subseteq X_0 \Delta X_1$.

Let us define $\overline{Y_0} = \{e \,:\, e \in E_0 \cap \overline{Y}\}$ and $\overline{Y_1} = \{e \,:\, e \in E_1 \cap \overline{Y}\}$.

Clearly $\overline{Y_0} \subseteq \overline{X_0}/S$ and $\overline{Y_1} \subseteq \overline{X_1}/S$ and $Y_0|_S = Y_1|_S = \phi$. Also $Y_0 \subseteq X_0 \;\Rightarrow\; Y_0 \in \mathcal{I}_0$ and $Y_1 \subseteq X_1 \;\Rightarrow\; Y_1 \in \mathcal{I}_1$. So $Y = Y_0 \Delta Y_1 \in \mathcal{I}$. Hence $\mathcal{I}$ satisfies (D2).

Let X and Y be two distinct disets belonging to $\mathcal{I}$, and be non-cancelling with $|X| = |Y| + 1$.

We will show that there exists $e \in X/Y$ such that $Y + e \in \mathcal{I}$.

Since $X,\ Y \in \mathcal{I} \;\Rightarrow$ there exists $X_0,\ X_1,\ Y_0,\ Y_1$ where $X_0,\ Y_0 \in \mathcal{I}_0$ and $X_1,\ Y_1 \in \mathcal{I}_1$ such that $X = X_0 \Delta X_1$ and $Y = Y_0 \Delta Y_1$ with $X_0|_S = X_1|_S$ and $Y_0|_S = Y_1|_S$.

Without any loss of generality, let $X_0|_S = X_1|_S = \phi$ and $Y_0|_S = Y_1|_S = \phi$.

Since X and Y are non-cancelling, X_0 is non-cancelling to Y_0 and X_1 is non-cancelling to Y_1. Again $|X| = |Y| + 1$, implies either $|X_0| > |Y_0|$ or $|X_1| > |Y_1|$.

For definiteness, let $|X_0| > |Y_0|$.

Then there exists $X_0' \subseteq X_0$ such that $|X_0'| = |Y_0| + 1$, so by (D3) there exists $e \in X_0'/Y_0$ such that $Y_0' = Y_0 + e \in \mathcal{I}_0$, and $Y_0'|_S = Y_1|_S = \phi$ and $Y_1 \in \mathcal{I}_1$

$\Rightarrow \quad Y' = Y + e = Y_0' \Delta Y_1 \in \mathcal{I}$, where $e \in X/Y$.

Thus $\mathcal{I}$ satisfies (D3). $\square$

Let h_0 and h_1 be the rank functions of D_0 and D_1 respectively and h be defined as

$$h(A) = h_0(A_0) + h_1(A_1) \quad \text{for all } A \in D(E),$$

where A_0 and A_1 are subsets of A and $\overline{A_0} = \overline{A} \cap E_0$ and $\overline{A_1} = \overline{A} \cap E_1$. We will show that h is the rank function of $D = (E, \mathcal{I})$.

Proof. (i) Clearly $h(\phi) = h_0(\phi_0) + h_1(\phi_1) = 0$.

(ii) Let $Y \subseteq X$, and X, $Y \in D(E)$, we have to show that $h(Y) \leq h(X)$.

Since X, $Y \in D(E)$, there exist X_0, X_1, Y_0, Y_1 such that

$$Y_0 \subseteq X_0 \quad Y_1 \subseteq X_1$$
$$\text{and } X = (X_0 \Delta X_1) = (X_0 \cup X_1), \quad Y = (Y_0 \Delta Y_1) = (Y_0 \cup Y_1)$$
$$\text{where } X_0,\ Y_0 \in D(E_0/S) \text{ and } X_1,\ Y_1 \in D(E_1/S).$$

So $h(Y) = h_0(Y_0) + h_1(Y_1) \leq h_0(X_0) + h_1(X_1) = h(X)$.

(iii) We will prove that, h is bisubmodular on $D(E)$. For X, $Y \in D(E)$

$$\begin{aligned} h(X) + h(Y) &= h_0(X_0) + h_1(X_1) + h_0(Y_0) + h_1(Y_1) \\ &\geq h_0(X_0 \cup Y_0) + h_0(X_0 \cap Y_0) + h_1(X_1 \cup Y_1) + h_1(X_1 \cap Y_1). \end{aligned} \tag{2.7.4}$$

Now

$$\begin{aligned} X \cup Y &= (X_0 \cup X_1) \cup (Y_0 \cup Y_1) \\ &= (X_0 \cup Y_0) \cup (X_1 \cup Y_1) \\ \text{and } X \cap Y &= (X_0 \cup X_1) \cap (Y_0 \cup Y_1) \\ &= (X_0 \cap Y_0) \cup (X_1 \cap Y_1). \end{aligned}$$

So,

$$h(X \cup Y) + h(X \cap Y) = h(X_0 \cup Y_0) + h(X_1 \cup Y_1) + h(X_0 \cap Y_0) + h(X_1 \cap Y_1). \tag{2.7.5}$$

(2.7.4) and (2.7.5) gives $h(X) + h(Y) \geq h(X \cup Y) + h(X \cap Y)$. Hence h is a bisubmodular function.

(iv) From the definition of h, $h(X) \leq |X|$, for all $X \in D(E)$.

Hence the proof. □

2.7.2 Perfect Ditroids

Definition 2.7.1 [44]. A ditroid $D = (E,\mathcal{I})$ is a perfect ditroid if and only if, its base family $\mathcal{B}$ is of the form

$$\mathcal{B} = \{B = (B_1, B_1^c) \quad \text{for some } B_1 \subseteq E\}.$$

We will now give two examples of perfect ditroids.

Example 2.7.1 Let $M = (E,\mathcal{J})$ be a matroid and B^m be any base of M. Define

$$\mathcal{I} = \{(A_1, A_2) \ : \ A_1 \subseteq B^m \text{ and } A_2 \subseteq E/B^m\}.$$

We will show that $D = (E,\mathcal{I})$, where $\mathcal{I}$ is defined as above, is a ditroid.

Proof. Clearly $\phi \in \mathcal{I}$. Since $\phi \subseteq B^m$ and $\phi \subseteq E/B^m \Rightarrow \phi = (\phi,\phi) \in \mathcal{I}$.

Let $Y \subseteq X$ and $X \in \mathcal{I}$, we will show that $Y \in \mathcal{I}$. Since, $X = (X_1, X_2) \in \mathcal{I}$ implies there exists a base B^m of M such that

$$\begin{aligned}
&X_1 \subseteq B^m \text{ and } X_2 \subseteq E/B^m.\\
&\text{And } \ Y = (Y_1, Y_2) \subseteq X = (X_1, X_2)\\
&\Rightarrow \ Y_1 \subseteq X_1 \text{ and } Y_2 \subseteq X_2\\
&\Rightarrow \ Y_1 \subseteq B^m \text{ and } Y_2 \subseteq E/B^m\\
&\Rightarrow \ Y = (Y_1, Y_2) \in \mathcal{I}.
\end{aligned}$$

Hence $\mathcal{I}$ satisfies (D2).

Let X and Y belong to $\mathcal{I}$ be non-cancelling with $|X| = |Y| + 1$. Since $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ belong to $\mathcal{I}$, there exist two bases B_1^m and B_2^m of M such that $X_1 \subseteq B_1^m$ and $X_2 \subseteq E/B_2^m$ and $Y_1 \subseteq B_2^m$ and $Y_2 \subseteq B_2^m$. Now, let $e \in X/Y$.

case (1) $e \in X_2$. Add e to Y_2 and let $Y_2' = Y_2 + e$. If $e \notin B_2^m$ then $Y_1 \subseteq B_2^m$ and $Y_2' \subseteq E/B_2^m$

$$\Rightarrow \qquad Y' = (Y_1, Y_2') \in \mathcal{I} \text{ and } |X| = |Y'|.$$

But if $e \in B_2^m$, then $B_3^m = (B_2^m/e + e')$ is a base of M, for some $e' \in E/B_2^m$, and $Y_1 \subseteq B_3^m$ and $Y_2' = Y_2 + e \subseteq E/B_3^m$

$\Rightarrow \quad Y' = (Y_1, Y_2') \in \mathcal{I}$.

Hence $\mathcal{I}$ satisfies (D3) in case-1.

case (2) When $e \in X_1$ and there is no $e' \in X/Y$ such that $e' \in X_2$.

Since X and Y are non-cancelling, this implies that $X_2 \subseteq Y_2$ that is, $|X_2| \leq |Y_2|$ and $|X_1| > |Y_1|$.

And $X_1, Y_1 \in \mathcal{J}$, so by the augmentation axiom of matroids there exists $e \in X_1/Y_1$ such that $Y_1' = Y_1 + e \in \mathcal{J}$. Now, it is enough to show that Y_1' is a subset of some base of M and Y_2 is a subset of the compliment of the same base.

If $e \in B_2^m$, then $Y_1' = Y_1 + e \subseteq B_2^m$ and we are done.

Let $e \notin B_2^m$, then there exists a circuit C of M contained in $B_2^m + e$. And there exists an $e' \in C$ such that $e' \notin Y_1$, otherwise $Y_1 + e$ will not be an independent set of the matroid M. If $B_3^m = (B_2^m + e)/e'$ then B_3^m is a base of M and $Y_1 + e \subseteq B_3^m$ and $Y_2 \subseteq E/B_3^m$.

So

$$Y' = (Y_1 + e, Y_2) \in \mathcal{I} \quad \text{and} \quad |X| = |Y'|.$$

Hence $\mathcal{I}$ satisfies (D3). This shows that $D = (E,\mathcal{I})$ is a ditroid. □

From the above discussion, we can say that if $X = (X_1, X_2) \in \mathcal{I}$ then $X_1 \in \mathcal{J}$ and $X_2 \in \mathcal{J}^*$ where $M^* = (E, \mathcal{J}^*)$ is the dual of $M = (E, \mathcal{J})$.

Let r and r^* be the rank functions respectively of M and M^*, and h be defined as

$$h(X) = h(X_1, X_2) = r(X_1) + r^*(X_2), \tag{2.7.6}$$

where r and r^* satisfy the following relationship [53],

$$r^*(A_1) = |A_1| - r(E) + r(A_1^c). \tag{2.7.7}$$

Theorem 2.7.3 h as defined in (2.7.6) satisfies the ditroid rank theorem, for the ditroid $D = (E, \mathcal{I})$, defined in example 2.7.1.

Proof. $h(\phi) = h(\phi, \phi) = r(\phi) + r^*(\phi) = 0$.

Let $Y = (Y_1, Y_2) \subseteq X = (X_1, X_2)$. Since $Y_1 \subseteq X_1$ $Y_2 \subseteq X_2$

$$h(X) = h(X_1, X_2) = r(X_1) + r^*(X_2) \geq r(Y_1) + r^*(Y_2) = h(Y_1, Y_2) = h(Y).$$

Also

$$h(X) = h(X_1, X_2) = r(X_1) + r^*(X_2) \leq |X_1| + |X_2| = |X_1 \cup X_2| = |X|.$$

We show that h is bisubmodular on $D(E)$.

Let $X = (X_1, X_2)$, $Y = (Y_1, Y_2) \in D(E)$, then using the relation (2.7.7) we have

$$\begin{aligned} h(X) + h(Y) &= h(X_1, X_2) + h(Y_1, Y_2) \\ &= r(X_1) + r^*(X_2) + r(Y_1) + r^*(Y_2) \\ &\geq r(X_1 \cup Y_1) + r(X_1 \cap Y_1) + |X_2| - r(E) + r(X_2^c) + \\ &\qquad |Y_2| - r(E) + r(Y_2^c) \\ &\geq r(X_1 \cup Y_1) + r(X_1 \cap Y_1) + |X_2 \cup Y_2| + |X_2 \cap Y_2| - 2r(E) + \\ &\qquad r(X_2^c \cup Y_2^c) + r(X_2^c \cap Y_2^c). \end{aligned} \tag{2.7.8}$$

Again we have,

$$\begin{aligned} h(X \cup Y) + h(X \cap Y) &= h((X_1 \cup Y_1)/(X_2 \cup Y_2), (X_2 \cup Y_2)/(X_1 \cup Y_1)) + \\ &\qquad h((X_1 \cap Y_1), (X_2 \cap Y_2)) \\ &= r((X_1 \cup Y_1)/(X_2 \cup Y_2)) + r^*((X_2 \cup Y_2)/(X_1 \cup Y_1)) + \\ &\qquad r(X_1 \cap Y_1) + r^*(X_2 \cap Y_2) \\ &\leq r(X_1 \cup Y_1) + r(X_1 \cap Y_1) + r^*(X_2 \cup Y_2) + r^*(X_2 \cap Y_2) \\ &= r(X_1 \cup Y_1) + r(X_1 \cap Y_1) + |(X_2 \cup Y_2)| - r(E) + \\ &\qquad r((X_2 \cup Y_2)^c) + |(X_2 \cap Y_2)| - r(E) + r((X_2 \cap Y_2)^c) \\ &= r(X_1 \cup Y_1) + r(X_1 \cap Y_1) + |(X_2 \cup Y_2)| + |(X_2 \cap Y_2)| - \\ &\qquad 2r(E) + r(X_2^c \cap Y_2^c) + r(X_2^c \cup Y_2^c). \end{aligned} \tag{2.7.9}$$

So from (2.7.8) and (2.7.9) we see that h is bisubmodular on $D(E)$. Hence h satisfies the ditroid rank theorem. □

Lemma 2.7.1 For the ditroid D defined in example (2.7.1), if $B = (B_1, B_2)$ is a base of D then B_1 is a base of M, and $B_2 = E/B_1$ is a base of M^*.

Proof. Let B be a base of D, and suppose B_1 is not a base of M.

Since $B = (B_1, B_2)$, implies there exists a base B_1^m of M such that $B_1 \subseteq B_1^m$ and $B_2 \subseteq E/B_1^m$.
Since B_1 is not a base of M, implies $|B_1| < |B_1^m|$, so we can augment B_1 to B_1' such that $B_1' \subseteq B_1^m$ and $B_2 \subseteq E/B_1^m$
$\Rightarrow \quad B' = (B_1', B_2) \in \mathcal{I}$ and $|B'| = |B| + 1$.
Which contradicts the fact that, B is a base of D. Hence the lemma. □

Thus, we conclude, that the base family of the above ditroid is of the form

$$\mathcal{B} = \{(B, B^c) \ : \ B \in \mathcal{J} \text{ and } B \text{ is a base of } M \ \},$$

since all the bases of M have the same cardinality, all the bases of D also have the same cardinality, and a base B of D has the form (B_1, B_1^c) for some B_1, base of M.

Example 2.7.2 Given a matroid $M = (E, \mathcal{J})$, it is possible to obtain another perfect ditroid, which has the perfect ditroid of example (2.7.1) as its sub-family. Define

$$\mathcal{I} = \{(A_1, A_2) \ : \ A_1 \subseteq I \in \mathcal{J} \text{ and } A_2 \subseteq E/I\}.$$

Then $D = (E, \mathcal{I})$ is a ditroid.

Proof. To show that the ditroid axioms (D1) and (D2) hold, the proof is same as in example (2.7.1). We need only to prove that the ditroid axiom (D3) is satisfied.

Let X, $Y \in \mathcal{I}$ be non-cancelling with $|X| = |Y| + 1$. There exist I^1 and I^2 belonging to $\mathcal{J}$ such that $X_1 \subseteq I^1, X_2 \subseteq E/I^1, Y_1 \subseteq I^2$ and $Y_2 \subseteq E/I^2$. Let $e \in X/Y$.

case (1) $e \in X_2$.
Since $e \notin Y_1$, we can add e to Y_2, and if $e \notin I^2$, then $Y_1 \subseteq I^2$ and $Y_2 + e \subseteq E/I^2$ and axiom (D3) holds.
But if $e \in I^2$, then choose $I^3 = I^2/e$ and $Y_1 \subseteq I^3$, $Y_2 \subseteq E/I^3$, and again (D3) holds.
Thus Y can always be augmented by adding an element from X_2.

case (2) There is no $e \in X/Y$ such that $e \in X_2$.
Given X and Y are non-cancelling this implies that
$X_2 \subseteq Y_2 \quad \Rightarrow \quad |X_1| > |Y_1|$.

Thus there exists $e \in X_1/Y_1$ such that $Y_1 + e \in \mathcal{J}$. Let $I^3 = Y_1 + e$. Then since $Y_2 \subseteq E/I^3$, $(Y_1 + e, Y_2) \in \mathcal{I}$.
Hence $D = (E, \mathcal{I})$ is a ditroid. □

Let h be the rank function of D and define it in the following way.

$$h(X) = h(X_1, X_2) = \max\{|Y_1 \cup Y_2| \ : \ (Y_1, Y_2) \subseteq (X_1, X_2) \text{ and } (Y_1, Y_2) \in \mathcal{I}\}.$$

If r is the rank function of M, then the relation between r and h is the following:

$$h(X) = h(X_1, X_2) = r(X_1) + |X_2|.$$

It can be easily verified that h satisfies the ditroid rank theorem for $D = (E, \mathcal{I})$, since h is the sum of a submodular and a modular function.
Base family $\mathcal{B}$ of the above ditroid $D = (E, \mathcal{I})$ is of the form :

$$\mathcal{B} = \{(I, I^c) \ : \ I \in \mathcal{J}\}.$$

Hence the ditroid $D = (E, \mathcal{I})$ is a perfect ditroid.

2.7.3 Dual of a Ditroid

Let $D = (E, \mathcal{I})$ be a ditroid. We define $\mathcal{I}^*$ as follows:

$$\mathcal{I}^* = \{X^* = (X_2, X_1) \ : \ X = (X_1, X_2) \in \mathcal{I}\}.$$

Theorem 2.7.4 $D^* = (E, \mathcal{I}^*)$ is a ditroid, and we call this ditroid as the dual ditroid of D.

Proof. Since $\phi = (\phi, \phi) \in \mathcal{I}$ by definition $\phi = (\phi, \phi) \in \mathcal{I}^*$. Hence $\mathcal{I}^*$ satisfies (D1).
Let $Y^* \subseteq X^*$ for $X^* \in \mathcal{I}^*$. We have to show that $Y^* \in \mathcal{I}^*$. Let $Y^* = (Y_2, Y_1)$.
$X^* = (X_2, X_1) \in \mathcal{I}^*$, for $X = (X_1, X_2) \in \mathcal{I}$, and $Y^* = (Y_2, Y_1) \subseteq X^*$ implies
$Y = (Y_1, Y_2) \subseteq X$. Since $(Y_1, Y_2) \in \mathcal{I}$, $Y^* = (Y_2, Y_1) \in \mathcal{I}^*$.

Let X^* and Y^* be two non-cancelling disets of $\mathcal{I}^*$, with $|X^*| = |Y^*| + 1$. Since X and Y satisfy (D3), it is evident that X^* and Y^* will also satisfy (D3). Hence $D^* = (E, \mathcal{I}^*)$ is a ditroid. □

Let h and h^* be the rank functions of D and D^* respectively. Then $h^*(X_2, X_1) = h(X_1, X_2)$.

Proof.

$$\begin{aligned} h^*(X^*) = h^*(X_2, X_1) &= \max\{|Y^*| \;:\; Y^* \subseteq X^* \text{ and } Y^* \in \mathcal{I}^*\} \\ &= \max\{|Y_2 \cup Y_1| \;:\; (Y_2, Y_1) \subseteq X^* \text{ and } (Y_2, Y_1) \in \mathcal{I}^*\} \\ &= \max\{|Y_1 \cup Y_2| \;:\; (Y_1, Y_2) \subseteq (X_1, X_2) \text{ and } (Y_1, Y_2) \in \mathcal{I}\} \\ &= \max\{|Y| \;:\; Y \subseteq X \text{ and } Y \in \mathcal{I}\} \\ &= h(X_1, X_2) = h(X). \end{aligned}$$

So, h^* satisfies the ditroid rank theorem. □

By definition of the dual ditroid D^* it is evident that dual of $D^* = (E, \mathcal{I}^*)$ is again $D = (E, \mathcal{I})$, i.e., $D^{**} = D$.

2.8 Relation Between Ditroids and Some Other Subset Systems

In this section, we relate ditroids with subset systems, like symmetric matroids, oriented matroids, generalised matroids and linking systems.

2.8.1 Pseudomatroids and Ditroids

Qi [44], established the relation between ditroids and pseudomatroids. He showed that, there is a one-to-one correspondence between pseudomatroids $P = (E, \mathcal{F})$ and perfect ditroids $D = (E, \mathcal{I})$, where $\mathcal{I}$ is defined as :

$$\mathcal{I} = \{(X, Y) \;:\; X \subseteq F, Y \subseteq E/F \text{ for some } F \in \mathcal{F}\}. \tag{2.8.1}$$

For the pseudomatroid $P = (E, \mathcal{F})$, the rank function b is defined as [7],

$$b(X, Y) = \max\{|X \cap F| - |Y \cap F| \;:\; F \in \mathcal{F}\}. \tag{2.8.2}$$

The rank function h of the corresponding perfect ditroid D is given by

$$h(X,Y) = b(X,Y) + |Y| = b^*(Y,X) + |X|, \tag{2.8.3}$$

where b^* is the rank function of the dual pseudomatroid of $P = (E,\mathcal{F})$.

It can be shown that generalised matroids and linking systems are also pseudomatroids, hence the corresponding perfect ditroids can be obtained from these subset systems too.

2.8.2 Symmetric Matroids And Ditroids

Consider a finite set E, and a partition π of E into 2-element classes, called the symmetric pairs. Any subset $T \subseteq E$ will be called a transversal (subtransversal) of π, if it contains precisely one (at most one) element of each symmetric pair. Two subtransversals T_1 and T_2 will be compatible if $T_1 \cup T_2$ is also a subtransversal. A transversal system with respect to π is a subset system $E' = (E,\mathcal{T})$ whose feasible sets are transversals of π. And

$$\hat{E}' = \{A \subseteq E \ : \ A \subseteq T \text{ for some } T \in \mathcal{T}\}. \tag{2.8.4}$$

Definition 2.8.1 The transversal system E' is a symmetric matroid, if the independence system $\hat{E}' \cap T$ is a matroid for every transversal $T \in \mathcal{T}$.

For each pair in π, order the elements as first and second. Now for each set $\overline{A} \in \hat{E}'$, we can define a diset (A_1, A_2), where the elements in A_1 are coming from the first element class and that of A_2 belong to the second element class of π.
Define

$$\mathcal{I} = \{(A_1, A_2) \ : \ \overline{A} = A_1 \cup A_2 \in \hat{E}'\}. \tag{2.8.5}$$

Then $D = (E,\mathcal{I})$ is a ditroid.
This can be shown by using the following lemma in [43].

Lemma 2.8.1 The system $D = (E,\mathcal{I})$ is a ditroid if and only if for any n dimensional $\{1,-1\}$ vector u, $M(u) = (E,\mathcal{I}(u))$, where $\mathcal{I}(u) = \{\overline{X} \ : \ X \in \mathcal{I},\ X \subseteq u\}$ forms a matroid.

We prove that $D = (E,\mathcal{I})$ is a ditroid in the following way.

Proof. $(\phi, \phi) \in \mathcal{I}$.
Let $(Y_1, Y_2) \subseteq (X_1, X_2)$ and $X = (X_1, X_2) \in \mathcal{I}$. Since $X \in \mathcal{I}$, $\overline{X} \in \widehat{E}'$. And $Y \subseteq X$, implies $\overline{Y} \in \widehat{E}'$. This Implies $Y = (Y_1, Y_2) \in \mathcal{I}$.

If (Y_1, Y_2) and $(X_1, X_2) \in \mathcal{I}$ are non-cancelling and $|(X_1, X_2)| = |(Y_1, Y_2)| + 1$, there exists an $e \in \overline{X}/\overline{Y}$ such that $\overline{Y} + e \in \widehat{E}'$, implies $Y + e \in \mathcal{I}$ where $e \in X/Y$. Thus $D = (E, \mathcal{I})$ is a ditroid. Hence the lemma. □

2.8.3 Oriented Matroids and Ditroids

Bland and Vergnas [3], developed the theory of Oriented matroids.

Let E be a finite set and Θ be a set of directed subsets of E. The subset system $M' = (E, \Theta)$ will be a **oriented matroid**, if Θ satisfies the following properties.

(1) For all $X \in \Theta$, $X \neq \phi$ implies $-X \in \Theta$; and for all X, $Y \in \Theta$. $\overline{Y} \subseteq \overline{X}$ implies $X = \pm Y$.

(2) For all $X = (X_1, X_2)$ and $(Y_1, Y_2) \in \Theta$, such that $X \neq -Y$ and for all $e \in (X_1 \cap Y_2) \cup (X_2 \cap Y_1)$ and $e' \in (X_1/Y_2) \cup (X_2/Y_1)$, there exists $Z = (Z_1, Z_2) \in \Theta$, such that

$$Z_1 \subseteq (X_1 \cup Y_1)/e, \qquad Z_2 \subseteq (X_2 \cup Y_2)/e \text{ and } e' \in \overline{Z}. \tag{2.8.6}$$

If $M' = (E, \Theta)$ is an oriented matroid with Θ as the collection of its circuits, then the unsigned circuits of Θ, denoted by $\overline{\Theta}$, define a **unique** matroid associated with the oriented matroid (E, Θ). That is

$$\mathcal{I} = \{I \ : \ C \not\subseteq I \text{ for all } C \in \Theta\}, \tag{2.8.7}$$

is the set of independent sets of (E, Θ), and

$$\overline{\mathcal{I}} = \{\overline{X} : X \in \mathcal{I}\},$$

are the independent sets of the matroid $(E, \overline{\Theta})$. Note that if $C \in \Theta$ then $-C \in \Theta$. If r' and r denote the rank functions of (E, Θ) and $(E, \overline{\Theta})$ respectively, then

$$r'(X_1, X_2) = \max\{|Y_1 \cup Y_2| \ : \ (Y_1, Y_2) \subseteq (X_1, X_2),\ (Y_1, Y_2) \in \mathcal{I}\} = r(X_1 \cup X_2).$$

Since $(E,\overline{\Theta})$ is a matroid, r is submodular. Hence it can be shown that r' is bisubmodular. We now show that $D = (E,\mathcal{I})$ satisfies the ditroid axioms.

By definition of $\mathcal{I}$, (D1) and (D2) are satisfied.

To show that the third axiom (D3) is also satisfied, let X, $Y \in \mathcal{I}$ be non-cancelling and $|Y| = |X| + 1$.

If $X + e \notin \mathcal{I}$ for all $e \in Y/X$, this implies that $\overline{X+e} \notin \overline{\mathcal{I}}$, for all $e \in \overline{Y}/\overline{X}$.

This is a contradiction, since $M = (E,\overline{\Theta})$ is a matroid and $\overline{X}$, $\overline{Y} \in \overline{\mathcal{I}}$. Thus $D = (E,\mathcal{I})$ is a ditroid.

In fact oriented matroids are self-dual ditroids.

Chapter 3

BISUBMODULAR SYSTEMS

3.1 Introduction

We devote this chapter to the study of bisubmodular systems. Polyhedral characterization of a bisubmodular system and some other bisubmodular polyhedra; like ditroid polyhedron, g-polymatroid etc. are discussed in the second section. Condition for a bisubmodular polyhedron to be non-empty is given and different operations on bisubmodular polyhedron, are discussed which help to generate new bisubmodular polyhedra.

In section three, the formal definition of generalized greedy algorithm (gga) is given and the generalized greedy solution (ggs) of an LPP with respect to a bisubmodular polyhedron is obtained. We show that all bisubmodular polyhedra have the total dual integrality property, and in case the function f defining the bisubmodular polyhedron, is integer valued than all extreme points of the polyhedron are also integer vectors. Greedy solutions of the other bisubmodular polyhedra are discussed in section four.

In section five, we briefly review the results on jump systems and their relationship with bisubmodular polyhedron from [6]. We then consider the $(0, \pm 1)$ extreme point bisubmodular polyhedra and define greedy systems as the collection of $(0, \pm 1)$ vectors of such polyhedra and show that greedy systems satisfy a 2-augmentation property and that ditroids are greedy

systems.

3.2 Bisubmodular Polyhedron

Let $E = \{e_1, e_2, \ldots, e_n\}$ be a finite set with cardinality n, and $D(E)$, denote the collection of directed subsets (or, disets) of E, i.e.

$$D(E) = \{(X_1, X_2) \ : \ X_1 \subseteq E \ X_2 \subseteq E \ \text{ and } X_1 \cap X_2 = \phi\}. \tag{3.2.1}$$

We have already defined a function $f \ : \ D(E) \longrightarrow \Re$, to be a bisubmodular function if it satisfies for any (X_1, X_2), $(Y_1, Y_2) \in D(E)$, the following condition,

$$f(X_1, X_2) + f(Y_1, Y_2) \geq f(X_1 \cap Y_1, X_2 \cap Y_2) + f((X_1 \cup Y_1)/(X_2 \cup Y_2), (X_2 \cup Y_2)/(X_1 \cup Y_1)). \tag{3.2.2}$$

The bisubmodular polyhedron associated with f and $D(E)$ is the following :

$$\mathcal{P}_f = \{ x \in \Re^n \ : \ x(X_1, X_2) \leq f(X_1, X_2), \ \text{ for all } (X_1, X_2) \in D(E)\}. \tag{3.2.3}$$

These polyhedra were first introduced by Dunstan and Welsh [14].

3.2.1 Submodular Polyhedra and Polymatroids

It can be immediately seen that if f' is a submodular function on E and $f'(\phi) = 0$, then f defined by $f(A, \phi) = f'(A)$, for $A \subseteq E$ and $f(A, B) = \infty$, for $B \neq \phi$, is bisubmodular. Bisubmodular polyhedron $\mathcal{P}_f$ associated with f will reduces to

$$\{ x \in \Re^n \ : \ x(A) \leq f'(A), \ \text{ for all } A \subseteq E \}, \tag{3.2.4}$$

which is the submodular polyhedron associated with f'.

If f' is non-decreasing and submodular and if $f(A, \phi) = f'(A)$, for $A \subseteq E$ and $f(\phi, j) = 0$ for $j \in E$ and $f(A, B) = \infty$, otherwise, then f is again bisubmodular, and $\mathcal{P}_f$ reduces to

$$\{ x \in \Re^n_+ \ : \ x(A) \leq f'(A), \ \text{ for all } A \subseteq E \}, \tag{3.2.5}$$

which is the polymatroid associated with f'.

3.2.2 Base Polyhedron

The base polyhedron [26]

$$\mathcal{B}_b = \{\, x \in \Re^n \;:\; x(A) \leq b(A), \text{ for all } A \subseteq E,\;\; x(E) = b(E)\}, \tag{3.2.6}$$

is the same as $\mathcal{P}_f$, where

$$f(A,B) = b(A) + b(E/B) - b(E), \tag{3.2.7}$$

is again a bisubmodular function on $D(E)$.

3.2.3 g-polymatroids

Another, more general, class of bisubmodular polyhedra consists of Frank's [22] g-polymatroids, defined as,

$$Q = Q(p,b) = \{\, x \in \Re^n \;:\; -p(A) \leq x(A) \leq b(A);\ \text{ for all } A \subseteq E\,\}, \tag{3.2.8}$$

where p and b are submodular function on E, and p and b satisfy the relation

$$b(X) + p(Y) \geq b(X/Y) + p(Y/X) \;\text{ for all } X,\; Y \subseteq E. \tag{3.2.9}$$

Let us define

$$f(A,B) = b(A) + p(B)\ \forall\ A,\ B \subseteq E,\ \text{ and } A \cap B = \phi, \tag{3.2.10}$$

that is

$$f(A,\phi) = b(A) \text{ and } f(\phi,B) = p(B), \text{ if } b(\phi) = p(\phi) = 0. \tag{3.2.11}$$

Then f is bisubmodular, and $Q(p,b) = \mathcal{P}_f$.

Also if

$$\mathcal{F} = \{A \subseteq E \;:\; A \text{ is a generalised matroid}\}, \tag{3.2.12}$$

then $\mathcal{P}_f$ is the convex hull of the characteristic vectors of the subsets in $\mathcal{F}$.

3.2.4 Pseudomatroids

The concept of pseudomatroids was introduced by Chandrasekaran and Kabadi, [7]. They gave the polyhedral characterization of pseudomatroids as,

$$x(A,B) \leq b(A,B) \quad \text{for all } (A,B) \in D(E), \tag{3.2.13}$$

where $b : D(E) \rightarrow \Re$ is an integer valued, bisubmodular function on $D(E)$ satisfying the following conditions :

1. $b(\phi,\phi) = 0$
2. $b(e_i,\phi) = 0$ or 1 for all $e_i \in E$
3. $\{ A \subseteq C;\ D \subseteq B;\ (A,B),\ (C,D) \in D(E)\} \Rightarrow b(A,B) \leq b(C,D)$.

They showed that the extreme points of the polytope (3.2.13) are precisely the characteristic vectors of the independents sets of a pseudomatroid on E, whose rank function is b.

3.2.5 Perfectly Matchable Subgraph Polytope

Let $G = (V,E)$ be an undirected graph. Define

$$\mathcal{F} = \{S \ : \ S \subseteq V \text{ and } G < S > \text{ has a perfect matching}\}. \tag{3.2.14}$$

Then $(V,\mathcal{F})$ is a matching pseudomatroid, and its rank function ρ is given by

$$\rho(A,B) = \max_{X \in \mathcal{F}}\{|X \cap A| - |X \cap B|\}. \tag{3.2.15}$$

It can be easily shown that $\rho(A,B)$ satisfies the following conditions :

1. $\rho(\phi,\phi) = 0$
2. $\rho(e_i,\phi) = 0$ or 1 for all $e_i \in E$

3. $\{ A \subseteq C;\ D \subseteq B;\ (A,B),\ (C,D) \in D(E)\} \Rightarrow \rho(A,B) \leq \rho(C,D)$.

Hence the corresponding polytope

$$x(A,B) \leq \rho(A,B) \tag{3.2.16}$$

is a pseudomatroid polyhedron and is also called the perfectly matchable subgraph polytope [1]. It is the convex hull of the characteristic vectors of perfectly matchable subgraphs of $G = (V,E)$.

3.2.6 Degree Sequence Polytope

Peled and Srinivasan in [41] describe the degree sequence polytope in detail. We show here that the degree sequence polytope is a bisubmodular polyhedron, by showing that the function defining this polytope is bisubmodular on $D(V)$, where V is the node set.

Let $G = (V,E)$ be a simple graph, the convex hull D_n of the degree sequences of G is the solution set of :

$$\left.\begin{array}{c} x(A,B) \leq f(A,B) \text{ for all } (A,B) \in D(V) \\ x_v \geq 0 \text{ for all } v \in V \end{array}\right\} \tag{3.2.17}$$

where $f(A,B) = |A|(n-1-|B|)$ and $|V| = n$.

Theorem 3.2.1 f is a bisubmodular function on $D(V)$. i.e., for any two disets (A,B) and $(C,D) \in D(V)$

$$f(A,B) + f(C,D) \geq f((A,B) \cup (C,D)) + f((A,B) \cap (C,D)) \tag{3.2.18}$$

Proof. Let

$$\begin{array}{ll} |A \cap C| = \alpha & |B \cap D| = \beta \\ |A \cap D| = \gamma & |B \cap C| = \delta \\ |A| = \alpha + \gamma + a & |B| = \beta + \delta + b \\ |C| = \alpha + \delta + c & |D| = \beta + \gamma + d \end{array}$$

where $a = |A/\{(A \cap C) \cup (A \cap D)\}|$ $\quad b = |B/\{(B \cap C) \cup (B \cap D)\}|$
$c = |C/\{(C \cap A) \cup (C \cap B)\}|$ $\quad d = |D/\{(D \cap A) \cup (D \cap B)\}|$.

$$\begin{aligned}
f(A,B) + f(C,D) &= (n-1)\{|A| + |C|\} - |A||B| - |C||D| \\
&= (n-1)\{\alpha + \gamma + a + \alpha + \delta + c\} - (\alpha + \gamma + a)(\beta + \delta + b) - \\
&\qquad (\alpha + \delta + c)(\beta + \gamma + d) \\
&= (n-1)\{2\alpha + \gamma + a + \delta + c\} - (\alpha + \gamma + a)(\beta + \delta + b) - \\
&\qquad (\alpha + \delta + c)(\beta + \gamma + d).
\end{aligned}$$

And

$$\begin{aligned}
f((A,B) \cup (C,D)) &+ f((A,B) \cap (C,D)) \\
&= (n-1)\{|A \cup C/B \cup D| + |A \cap C|\} - \\
&\qquad (|A \cup C/B \cup D|)(|B \cup D/A \cup C|) - |A \cap C||B \cap D| \\
&= (n-1)\{\alpha + \gamma + a + \delta + c - \gamma - \delta + \alpha\} - \\
&\qquad (\alpha + a + c)(\beta + b + d) - \alpha\beta \\
&= (n-1)(2\alpha + a + c) - (\alpha + a + c)(\beta + b + d) - \alpha\beta.
\end{aligned}$$

(3.2.18) will be true if and only if,

$$\begin{aligned}
&(n-1)(2\alpha + a + c + \gamma + \delta) - (\alpha + \gamma + a)(\beta + \delta + b) - (\alpha + \delta + c)(\beta + \gamma + d) \\
&\qquad \geq (n-1)(2\alpha + a + c) - (\alpha + a + c)(\beta + b + d) - \alpha\beta
\end{aligned}$$

iff, $\quad (n-1)(\gamma + \delta) + \alpha\beta - (\alpha + \gamma + a)(\beta + \delta + b) - (\alpha + \delta + c)(\beta + \gamma + d) +$
$\qquad (\alpha + a + c)(\beta + b + d) \geq 0$

iff, $\quad (n-1)(\gamma + \delta) + \alpha\beta - \alpha\beta - \alpha\delta - b\alpha - \beta\gamma - \gamma\delta - b\gamma - a\beta - a\delta -$
$\qquad ab - \alpha\beta - \alpha\gamma - d\alpha - \beta\delta - \gamma\delta - d\delta - c\beta - c\gamma - cd +$
$\qquad\qquad \alpha\beta + b\alpha + d\alpha + a\beta + ab + ad + c\beta + bc + cd \geq 0$

iff, $\quad (n-1)(\gamma + \delta) - \alpha\delta - \alpha\gamma - \beta\gamma - 2\gamma\delta - a\delta - b\gamma - c\gamma - d\delta + ad + bc \geq 0$

iff, $\quad (n-1)(\gamma + \delta) - \alpha(\gamma + \delta) - \beta(\gamma + \delta) - (\gamma + \delta)^2 + \gamma^2 + \delta^2 - a(\gamma + \delta) -$
$\qquad b(\gamma + \delta) - c(\gamma + \delta) - d(\gamma + \delta) + a\gamma + b\delta + c\delta + d\gamma + ad + bc \geq 0$

iff, $\quad (\gamma + \delta)\{(n-1) - \alpha - \beta - \gamma - \delta - a - b - c - d\} + \gamma^2 + \delta^2 +$
$\qquad a\gamma + b\delta + c\delta + d\gamma + ad + bc \geq 0. \qquad (3.2.19)$

If $A \cup B \cup C \cup D \subset E$, then

$$(n-1) - \alpha - \beta - \gamma - \delta - a - b - c - d \geq 0$$

and since

$$\gamma^2 + \delta^2 + a\gamma + b\delta + c\delta + d\gamma + ad + bc \geq 0.$$

(3.2.19) is true.

If $A \cup B \cup C \cup D = E$, then

$$(n-1) - \alpha - \beta - \gamma - \delta - a - b - c - d = -1,$$

and (3.2.19) reduces to,

$$-(\gamma + \delta) + \gamma^2 + \delta^2 + a\gamma + b\delta + c\delta + d\gamma + ad + bc \geq 0.$$

But γ and δ are integers, thus $\gamma^2 > \gamma$ and $\delta^2 > \delta$ and (3.2.19) is true.

From the above theorem, we conclude that the polyhedron (3.2.17) is a bisubmodular polyhedron. □

In section 3.5.1 using 2-SA, we are able to give a simpler proof of the above result. In [12], Cunningham and Krotki define the b-matching $u-$capacitated degree sequence polyhedron and show that it is a bisubmodular polyhedron.

3.2.7 Ditroid Polyhedra

In section (2.4), we proved that if $D = (E, \mathcal{I})$ is a ditroid, then h, its rank function is bisubmodular. The associated polyhedron $\mathcal{P}_h$ is therefore, a bisubmodular polyhedron. Latter we will show that $\mathcal{P}_h$ is the convex hull of the characteristic vectors of the subsets in $\mathcal{I}$.

Definition 3.2.1 For a vector $x \in \mathcal{P}_f$, we say that a diset (A, B) is x-tight (or, tight) if $x(A, B) = f(A, B)$.

Chandrasekaran & Kabadi proved the following lemma.

We will show that the condition is also sufficient by using induction on $|E|$.
For $|E| = 1$,

$$f(e,\phi) + f(\phi,e) \geq 0$$

$$\Rightarrow \qquad f(e,\phi) \geq -f(\phi,e).$$

Choose $x(e) \in (-f(\phi,e), f(e,\phi))$, then $x(e) \in \mathcal{P}_f$. Hence the condition is sufficient for $|E| = 1$.

Now suppose the theorem is true for $k = |E| - 1$. Let $s \in E$ and $E^1 = E/s$. Define $f_1 = f|_{E^1}$.
Since f is a bisubmodular function on $D(E)$, f_1 its restriction to E^1 is also a bisubmodular function on $D(E^1)$. Hence $\mathcal{P}_{f_1}$ is a bisubmodular polyhedron, and since the result is true for $k = |E^1|$ there exists a vector $x_1 \in \Re^{|E^1|}$, such that $x_1 \in \mathcal{P}_{f_1}$.
Let

$$m = \min_{(A,B)\in D(E)} \{f(A,B) - x_1(A,B) \;:\; s \in A\}.$$

and

$$M = \max_{(C,D)\in D(E)} \{x_1(C,D) - f(C,D) \;:\; s \in D\}.$$

We claim that $\qquad m \geq M.$
$s \in A \cap D$, and

$$\begin{aligned} f(A,B) + f(C,D) &\geq f(A\cup C/B\cup D, B\cup D/A\cup C) + f(A\cap C, B\cap D) \\ &\geq x_1(A\cup C/B\cup D, B\cup D/A\cup C) + x_1(A\cap C, B\cap D) \\ &= x_1(A/s, B) + x_1(C, D/s) \\ \Rightarrow f(A,B) - x_1(A/s,B) &\geq x_1(C,D/s) - f(C,D). \end{aligned} \tag{3.2.20}$$

$$\Rightarrow \qquad \min_{\substack{(A,B)\in D(E)\\ s\in A}} \{f(A,B) - x_1(A,B)\} \geq \max_{\substack{(C,D)\in D(E)\\ s\in D}} \{x_1(C,D) - f(C,D)\}$$

$$\Rightarrow \qquad m \geq M.$$

Now, define $x \in \Re^{|E|}$, such that $x|_{E^1} = x_1$ and $x(s) \in (M, m)$.

Thus there exists an $x \in \mathcal{P}_f$.

If f is integral, then m, M will be integers and hence $x(s)$ can be chosen an integer. Thus $\mathcal{P}_f$ contains integer points. □

Since $f(A, B) + f(B, A) \geq 2f(\phi, \phi)$, a sufficient condition for $\mathcal{P}_f$ to be non-empty is that $f(\phi, \phi) = 0$.

When f is a rank function of a matroid, a pseudomatroid or a ditroid the condition $f(\phi, \phi)$ is automatically satisfied, but when f is a defining function of a g polymatroid, than by definition, we have

$$f(A, B) = b(A) + p(B) \text{ and } f(\phi, \phi) = 0.$$

Bisubmodularity of f implies that b and p are fully submodular on subsets of E and that p and b are compliant follows from bisubmodularity of f and $f(\phi, \phi) = 0$. But (p, b) being a strong pair is a sufficient condition for $Q = (p, b)$ to be non-empty.

3.2.8 Different Operations On Bisubmodular Polyhedron

Bouchet and Cunningham [6] defined operations like reflection and restriction (or, projection) on a bisubmodular polyhedron and showed that these operations generate new bisubmodular polyhedra. We reproduce some of their results below.

Reflection

Let $S \subseteq E$. For each $x \in \Re^n$, let x' be a vector defined as follows :

$$\left.\begin{aligned} x'_j &= x_j \quad \text{if } e_j \notin S \\ x'_j &= -x_j \quad \text{if } e_j \in S \end{aligned}\right\} \tag{3.2.21}$$

then x' is known as the reflection of x with respect to S. The following lemma has been proved in [6].

Lemma 3.2.2 Let f be bisubmodular and $S \subseteq E$. Then reflecting $\mathcal{P}_f$ with respect to S, gives a bisubmodular polyhedron $\mathcal{P}_{f'}$, where f' is defined by

$$f'(A,B) = f((A/S) \cup (B \cap S), (B/S) \cup (A \cap S)), \;\; (A,B) \in D(E). \tag{3.2.22}$$

Restriction or Projection

Let $x \in \mathcal{P}_f$ and $S \subseteq E$, then define a vector x' as follows

$$x' = x|_S \text{ such that } x \in \mathcal{P}_f. \tag{3.2.23}$$

Lemma 3.2.3. $\mathcal{P}_{f'}$ is a bisubmodular polyhedron, where

$$\mathcal{P}_{f'} = \{x' \;:\; x' = x|_S \text{ and } x'(A,B) \leq f'(A,B)\}, \tag{3.2.24}$$

and f' is the restriction of f on S.

3.3 Greedy Solution and TDI of a Bisubmodular Polyhedron

Linear program associated with $\mathcal{P}_f$ is

$$\left.\begin{array}{c} \max\; cx \\ \text{subject to } \;\; x(A,B) \leq f(A,B) \;\; \text{for all } (A,B) \in D(E). \end{array}\right\} \tag{3.3.1}$$

The dual of (3.3.1) is

$$\min \sum f(A,B) z_{(A,B)} \tag{3.3.2a}$$

$$\text{subject to} \quad \sum_{\substack{(A,B) \in D(E) \\ e_i \in A}} z_{(A,B)} - \sum_{\substack{(A,B) \in D(E) \\ e_i \in B}} z_{(A,B)} = c_i \qquad \text{for all } e_i \in E \tag{3.3.2b}$$

$$\text{and } \; z_{(A,B)} \geq 0 \qquad \text{for all } (A,B) \in D(E). \tag{3.3.2c}$$

It is evident that (3.3.1) will have a finite optimal solution, if $f(A,B) < \infty$ for all $(A,B) \in D(E)$. In order that the constraint set be non-empty we also assume that $f(\phi, \phi) = 0$.

3.3.1 Generalised Greedy Algorithm (gga)

In [6] the idea of a greedy algorithm for bisubmodular systems was given. We extend the results in [54] for solving linear programs over polymatroids to develop a greedy algorithm for solving (3.3.1).

For the c given in (3.3.1), let

$$P(c) = \{\, i_1, i_2 \dots, i_n \;:\; |c_{i_1}| \geq |c_{i_2}| \geq \cdots \geq |c_{i_n}| \,\}.$$

$P(c)$ is the set of permutations with respect to c. For a $p \in P(c)$, we solve (3.3.1) in the following way, which is known as the generalized greedy algorithm (gga).
Define

$$\begin{aligned} A^0 &= B^0 = \phi \\ A^i = A^{i-1} &\text{ and } B^i = B^{i-1} + e_i \qquad \text{if } c_i \leq 0 \\ A^i = A^{i-1} + e_i &\text{ and } B^i = B^{i-1} \qquad \text{if } c_i > 0, \end{aligned}$$

where the indices i are ordered as in p, then

$$x_i = x(e_i) = \begin{cases} f(A^i, B^i) - f(A^{i-1}, B^{i-1}) & \text{if } \; c_i > 0 \\ f(A^{i-1}, B^{i-1}) - f(A^i, B^i) & \text{if } \; c_i \leq 0. \end{cases} \tag{3.3.3}$$

Claim : The resulting vector x is the **generalized greedy solution (ggs)**, of (3.3.1) via p.

Note that, this method does not require c_i's to be distinct.

3.3.2 Feasibility of x

Define

$$S^i = (A^i, B^i) = (S_1^i, S_2^i) \qquad \text{for all } i \text{ and } S^0 = \phi$$

By definition of S^n, $\overline{S^n} = E$, then

$$x_i = x(e_i) = \begin{cases} f(S^i) - f(S^{i-1}) & \text{if } \; c_i > 0 \\ f(S^{i-1}) - f(S^i) & \text{if } \; c_i \leq 0 \end{cases} \tag{3.3.4}$$

and,

$$\phi = S^0 \subset S^1 \subset S^2 \cdots \subset S^{n-1} \subset S^n.$$

Let $(C, D) \in D(E)$ and let $\{e_{l_1}, e_{l_2}, \dots, e_{l_p}\} = (C, D) \cap (S_1^n, S_2^n)$ and $\{e_{m_1}, e_{m_2}, \dots, e_{m_q}\} = (C, D) \cap (S_2^n, S_1^n)$, where $l_1 < l_2 \cdots < l_p$ and $m_1 < m_2 \cdots < m_q$. Here note that, $\{e_{l_1}, e_{l_2}, \dots, e_{l_p}\}$ and $\{e_{m_1}, e_{m_2}, \dots, e_{m_q}\}$ are both disets.

Now,

$$\begin{aligned}
x(C, D) &= \sum_{e_i \in C \cap S_1^n} x_i + \sum_{e_i \in C \cap S_2^n} x_i - \sum_{e_i \in D \cap S_1^n} x_i - \sum_{e_i \in D \cap S_2^n} x_i \\
&= \left\{ \sum_{e_i \in C \cap S_1^n} x_i - \sum_{e_i \in D \cap S_2^n} x_i \right\} + \left\{ \sum_{e_i \in C \cap S_2^n} x_i - \sum_{e_i \in D \cap S_1^n} x_i \right\} \\
&= [\{ f(S^{l_1}) - f(S^{l_1 - 1}) \} + \{ f(S^{l_2}) - f(S^{l_2 - 1}) \} + \\
&\qquad \cdots + \{ f(S^{l_p}) - f(S^{l_p - 1}) \}] + \\
&\qquad\quad [\{ f(S^{m_1 - 1}) - f(S^{m_1}) \} + \{ f(S^{m_2 - 1}) - f(S^{m_2}) \} + \\
&\qquad\qquad \cdots + \{ f(S^{m_q - 1}) - f(S^{m_q}) \}].
\end{aligned}$$

For feasibility of x, we have to show that

$$\begin{aligned}
&\{f(S^{l_1}) - f(S^{l_1-1}) + f(S^{l_2}) - f(S^{l_2-1}) + \cdots + f(S^{l_p}) - f(S^{l_p-1})\} \\
&\quad + \{f(S^{m_1-1}) - f(S^{m_1}) + f(S^{m_2-1}) - f(S^{m_2}) + \cdots + f(S^{m_q-1}) - f(S^{m_q})\} \le f(C, D)
\end{aligned}$$

or that,

$$\begin{aligned}
&f(S^{l_1}) + f(S^{l_2}) + \cdots + f(S^{l_p}) + f(S^{m_1-1}) + f(S^{m_2-1}) + \cdots + f(S^{m_q-1}) \le f(C, D) \\
&\quad + f(S^{l_1-1}) + f(S^{l_2-1}) + \cdots + f(S^{l_p-1}) + f(S^{m_1}) + f(S^{m_2}) + \cdots + f(S^{m_q}). \qquad (3.3.5)
\end{aligned}$$

Consider the minimum and the second minimum numbers of

$l_1 - 1,\ l_2 - 1, \dots, l_p - 1,\ m_1, m_2, \dots, m_q$.

Four cases are possible,

case (i)	minimum is	$l_1 - 1$	second minimum is	$l_2 - 1$,
case (ii)	minimum is	$l_1 - 1$	second minimum is	m_1,
case (iii)	minimum is	m_1	second minimum is	m_2,
case (iv)	minimum is	m_1	second minimum is	$l_1 - 1$.

n case (i),

$$\begin{aligned} f(C,D) &+ f(S^{l_1-1}) + f(S^{l_2-1}) \\ &\geq f(\phi,\phi) + f((C,D)\cup S^{l_1-1}) + f(S^{l_2-1}) \\ &= f((C,D)\cup S^{l_1}) + f(S^{l_2-1}) \\ &\geq f(((C,D)\cup S^{l_1})\cup S^{l_2-1}) + f(S^{l_1}) \\ &= f(S^{l_1}) + f((C,D)\cup S^{l_2}). \end{aligned} \tag{3.3.6}$$

n case (ii),

$$\begin{aligned} f(C,D) &+ f(S^{l_1-1}) + f(S^{m_1}) \\ &\geq f((C,D)\cup S^{l_1-1}) + f(S^{m_1}) \\ &= f((C,D)\cup S^{l_1}) + f(S^{m_1}) \\ &\geq f(((C,D)\cup S^{l_1})\cap S^{m_1}) + f(((C,D)\cup S^{l_1})\cup S^{m_1}) \\ &= f(S^{l_1}) + f(((C,D)\cup S^{l_1})\cup S^{m_1}) \\ &= f(S^{l_1}) + f(((C,D)/e_{m_1})\cup S^{m_1-1}). \end{aligned} \tag{3.3.7}$$

Similarly, for case (iii), we get

$$f(C,D) + f(S^{m_1}) + f(S^{m_2}) \geq f(S^{m_1-1}) + f((((C,D)/e_{m_1})/e_{m_2})\cup S^{m_2-1}), \tag{3.3.8}$$

and for case (iv), we have

$$f(C,D) + f(S^{m_1}) + f(S^{l_1-1}) \geq f(S^{m_1-1}) + f(((C,D)/e_{m_1})\cup S^{l_1}). \tag{3.3.9}$$

Suppose case (i) holds for this (C,D) and m_2 is the next minimum integer. Adding $f(S^{m_2})$ to both sides of (3.3.6) will give

$$\begin{aligned} f(C,D) + f(S^{l_1-1}) + f(S^{l_2-1}) + f(S^{m_2}) &\geq f(S^{l_1}) + f((C,D)\cup S^{l_2}) + f(S^{m_2}) \\ \Rightarrow f(C,D) + f(S^{l_1-1}) + f(S^{l_2-1}) + f(S^{m_2}) &\geq f(S^{l_1}) + f(S^{l_2}) + \\ & \qquad f((C,D)/e_{m_2}\cup S^{m_2-1}). \end{aligned}$$

Continuing this process we will arrive at the inequality (3.3.5), since the last term on the right hand side of the above inequality will either reduce to $f(S^l)$ or $f(S^{m-1})$. Hence the solution given in (3.3.4) is feasible.

3.3.3 Dual Solution and Its Feasibility

Let us define z as follows :

$$z_{S^i} = \begin{cases} c_i - c_{i+1}, & \text{if } e_i \in S_1^i \text{ and } e_{i+1} \in S_1^{i+1} \\ c_{i+1} - c_i, & \text{if } e_i \in S_2^i \text{ and } e_{i+1} \in S_2^{i+1} \\ -c_i - c_{i+1}, & \text{if } e_i \in S_2^i \text{ and } e_{i+1} \in S_1^{i+1} \\ c_i + c_{i+1}, & \text{if } e_i \in S_1^i \text{ and } e_{i+1} \in S_2^{i+1} \end{cases} \tag{3.3.10a}$$

for $i = 1, 2, \ldots (n-1)$.

And for $i = n$

$$z_{S^n} = \begin{cases} c_n & \text{if } e_n \in S_1^n \\ -c_n & \text{if } e_n \in S_2^n \end{cases} \tag{3.3.10b}$$

and $z_{(A,B)} = 0$ for all $(A, B) \neq S^i$ for all $i \in [1, n]$.

Since $S^1 \subset S^2 \subset \cdots \subset S^n$ form a chain in $D(E)$, the coefficient matrix of the constraints in the dual problem (3.3.2) with respect to z_{S^i}'s is an upper triangular matrix. To prove the feasibility of z, we notice that the n$^{\text{th}}$ dual constraint is satisfied by definition of z_{S^n}, i.e,

$$z_{S^n} = c_n \quad \text{if} \quad e_n \in S_1^n$$

$$\text{and} \qquad z_{S^n} = -c_n \quad \text{if} \quad e_n \in S_2^n.$$

For the (n - 1)$^{\text{th}}$ dual constraint, four cases are possible.

(i) $e_n, e_{n-1} \in S_1^n$.

(ii) $e_n, e_{n-1} \in S_2^n$.

(iii) $e_n \in S_1^n, \; e_{n-1} \in S_2^n$.

(iv) $e_n \in S_2^n, \; e_{n-1} \in S_1^n$.

case (i) In this case the nth and (n - 1)th dual constraints are

$$z_{S^{n-1}} + z_{S^n} = c_{n-1} \text{ and } z_{S^n} = c_n,$$

and $z_{S^{n-1}} = c_{n-1} - c_n$ and $z_{S^n} = c_n$, satisfy these constraints.

case (ii) Dual constraints are

$$-z_{S^{n-1}} - z_{S^n} = c_{n-1} \text{ and } -z_{S^n} = c_n,$$

and since $z_{S^{n-1}} = c_n - c_{n-1}$ and $z_{S^n} = -c_n$, these constraints are also satisfied. Similarly for cases (iii) and (iv), dual feasibility of the nth and (n - 1)th constraints can be verified.

Assuming that starting from n up to (n - k)th dual constraints are satisfied, we will now show that the (n - (k + 1))th dual constraint is also satisfied by the solution given in (3.3.10).

Corresponding to the signs of c_{n-k} and $c_{n-(k+1)}$, four cases are possible.

(i) $e_{n-k}, e_{n-(k+1)} \in S_1^n$.

(ii) $e_{n-k}, e_{n-(k+1)} \in S_2^n$.

(iii) $e_{n-k} \in S_1^n$, $e_{n-(k+1)} \in S_2^n$.

(iv) $e_{n-k} \in S_2^n$, $e_{n-(k+1)} \in S_1^n$.

case (i) The last $n-(k+1)$ dual constraints are

$$\sum_{j=0}^{k+1} z_{S^{n-j}} = c_{n-(k+1)} \tag{3.3.11}$$

$$\sum_{j=0}^{l} z_{S^{n-l}} = c_{n-l} \qquad \text{for } l = 0, 1, \ldots, k. \tag{3.3.12}$$

By our assumption constraints (3.3.12) are satisfied and (3.3.11) can be written as

$$\begin{aligned} \sum_{j=0}^{k+1} z_{S^{n-j}} &= \sum_{j=0}^{k} z_{S^{n-j}} + z_{S^{n-(k+1)}} \\ &= c_{n-k} + c_{n-(k+1)} - c_{n-k} = c_{n-(k+1)}. \end{aligned}$$

Hence the (n -(k + 1))th constraint is also satisfied. We can argue for the other three cases in a similar manner. Hence the dual solution z given in (3.3.10) is feasible.

3.3.4 Optimality of x

Without any loss of generality, let us assume that the first k_1 elements are forward elements of S^n, next k_2 elements are backward elements of S^n, and next k_3 elements are forward elements of S^n and so on, and finally k_l elements are forward elements of S^n. It is not necessary that all k_i's have to be non-zero.

The dual objective function value in this case is

$$\begin{aligned}
&= \sum_{i=1}^{n} f(S^i) z_{(S^i)} \\
&= \sum_{i=1}^{k_1} f(S^i) z_{(S^i)} + \sum_{i=k_1+1}^{k_2} f(S^i) z_{(S^i)} + \cdots + \sum_{i=k_{l-1}+1}^{k_l} f(S^i) z_{(S^i)} \\
&= \sum_{i=1}^{k_1-1} f(S^i)(c_i + c_{i+1}) + f(S^{k_1})(c_{k_1} + c_{k_1+1}) + \sum_{i=k_1+1}^{k_2-1} f(S^i)(c_{i+1} - c_i) + \\
&\qquad f(S^{k_2})(-c_i - c_{i+1}) + \cdots + \sum_{i=k_{l-1}+1}^{k_l-1} f(S^i)(c_i - c_{i+1}) + f(S^{k_l}) c_{k_l}. \qquad (3.3.13)
\end{aligned}$$

And the primal objective function value

$$\begin{aligned}
&= \sum_{i=1}^{n} c_i x_i \\
&= \sum_{i=1}^{k_1} c_i \{ f(S^i) - f(S^{i-1}) \} + \sum_{i=k_1+1}^{k_2} c_i \{ f(S^{i-1}) - f(S^i) \} + \\
&\qquad \cdots + \sum_{i=k_{l-1}+1}^{k_l} c_i \{ f(S^i) - f(S^{i-1}) \} \\
&= \sum_{i=1}^{k_1-1} c_i \{ f(S^i) - f(S^{i-1}) \} + c_{k_1} \{ f(S^{k_1}) - f(S^{k_1-1}) \} + \\
&\qquad \sum_{i=k_1+1}^{k_2-1} c_i \{ f(S^{i-1}) - f(S^i) \} + c_{k_2} \{ f(S^{k_2-1}) - f(S^{k_2}) \} + \\
&\qquad \cdots + \sum_{i=k_{l-1}+1}^{k_l} c_i \{ f(S^i) - f(S^{i-1}) \} \\
&= \sum_{i=1}^{k_1-1} f(S^i)(c_i - c_{i+1}) + c_{k_1} f(S^{k_1}) + c_{k_1+1} f(S^{k_1}) + \\
&\qquad \sum_{i=k_1+1}^{k_2-1} f(S^i)(c_{i+1} - c_i) - c_{k_2} f(S^{k_2}) - c_{k_2+1} f(S^{k_2}) + \\
&\qquad \cdots + \sum_{i=k_{l-1}+1}^{k_l-1} f(S^i)(c_i - c_{i+1}) + c_{k_l} f(S^{k_l})
\end{aligned}$$

$$
\begin{aligned}
= & \sum_{i=1}^{k_1-1} f(S^i)(c_i - c_{i+1}) + f(S^{k_1})(c_{k_1} + c_{k_1+1}) + \\
& \sum_{i=k_1+1}^{k_2-1} f(S^i)(c_{i+1} - c_i) + f(S^{k_2})(-c_{k_2} - c_{k_2+1}) + \\
& \cdots + \sum_{i=k_{l-1}+1}^{k_l-1} f(S^i)(c_i - c_{i+1}) + c_{k_l} f(S^{k_l}),
\end{aligned}
$$

which is equal to (3.3.13), that is the dual objective function value. Hence x is an optimal solution to problem (3.3.1), and z is the dual optimal solution.

In [43] Qi, posed the following question.
Define

$$
\left.\begin{aligned} g(Y) = \max\ \{Yx \ : \ x \in \mathcal{P}_f\} \\ g(\phi, \phi) = 0 \end{aligned}\right\}. \tag{3.3.14}
$$

Then $g(Y)$ is a set function on $D(E)$.
Under what condition on f is $g(Y)$ a bisubmodular function ?

By the generalised greedy algorithm discussed above, for f finite valued it follows that $g(Y)$ is bisubmodular and in fact

$$
g(Y) = f(Y) \text{ for all } Y \in D(E). \tag{3.3.15}
$$

Here the objective function vector $c = \chi_{Y(e_i)}$, is the characteristic vector of the diset Y.

Let $|Y| = l$. Choose a chain S, such that $\phi = S^0 \subset S^1 \subset S^2 \cdots \subset S^l = Y$ and $S^{l+1} \subset S^{l+2} \cdots \subset S^n = (Y_1, Y_2 \cup E/\overline{Y})$. That is the elements of $E/\overline{Y}$ are added one by one to $S^{l+1} \ldots S^n$ as backward elements. The dual optimal solution by (3.3.10) is

$$
\left.\begin{aligned} z_{S^l} &= 1 \\ z_T &= 0 \text{ for all } T(\neq S^l) \in D(E). \end{aligned}\right\} \tag{3.3.16}
$$

Thus

$$
g(Y) = f(S^l) = f(Y), \text{ for all } Y \in D(E).
$$

3.3.5 TDI of a Bisubmodular Polyhedron

If the objective function vector c of (3.3.1) is an integer vector i.e. $c_1, c_2, \ldots, c_n$ are all integers then from the dual solution (3.3.10) we note that each component of the dual solution z is integral. Hence (3.3.1) represents a total dual integral system.

Moreover, if f is an integral function then the primal solution (3.3.4) is also integral. Qi, [43]. has shown that the system of linear inequalities

$$x(T) \leq f(T) \text{ for all } T \in \mathcal{I}$$

is box total dual integral, where $\mathcal{I}$ is an intersecting family.

This implies the total dual integrality of (3.3.1). However we have given an alternate proof using the greedy algorithm.

3.4 Greedy Solutions of Some Other Bisubmodular Polyhedra

3.4.1 Ditroid Polyhedron

Let $D = (E, \mathcal{I})$ be a ditroid and let h denote its rank function. Consider the linear program

$$\left.\begin{array}{c} \max\ cx \\ \text{subject to} \quad x(A,B) \leq h(A,B) \quad \text{for all } (A,B) \in D(E). \end{array}\right\} \tag{3.4.1}$$

The dual of (3.4.1) can be written as follows :

$$\min \sum h(A,B) z_{(A,B)} \tag{3.4.2a}$$

$$\text{subject to} \quad \sum_{\substack{(A,B)\in D(E) \\ e_i \in A}} z_{(A,B)} - \sum_{\substack{(A,B)\in D(E) \\ e_i \in B}} z_{(A,B)} = c_i \quad \forall\ e_i \in E \tag{3.4.2b}$$

$$\text{and } z_{(A,B)} \geq 0 \qquad \forall\ (A,B) \in D(E). \tag{3.4.2c}$$

Clearly, the ggs of (3.4.1) is :

$$x_i = x(e_i) = \begin{cases} h(S^i) - h(S^{i-1}) & \text{if } c_i > 0 \\ h(S^{i-1}) - h(S^i) & \text{if } c_i \leq 0. \end{cases} \tag{3.4.3}$$

Since h is a ditroid rank function,

$$x_i = x(e_i) = \begin{cases} h(S^i) - h(S^{i-1}) & = 1 \text{ or } 0 & \text{if } c_i > 0 \\ h(S^{i-1}) - h(S^i) & = -1 \text{ or } 0 & \text{if } c_i \leq 0. \end{cases} \tag{3.4.4}$$

Similarly the solution to the dual problem (3.4.2) is

$$z_{S^i} = \begin{cases} c_i - c_{i+1}, & \text{if } e_i \in S_1^i \text{ and } e_{i+1} \in S_1^{i+1} \\ c_{i+1} - c_i, & \text{if } e_i \in S_2^i \text{ and } e_{i+1} \in S_2^{i+1} \\ -c_i - c_{i+1}, & \text{if } e_i \in S_2^i \text{ and } e_{i+1} \in S_1^{i+1} \\ c_i + c_{i+1}, & \text{if } e_i \in S_1^i \text{ and } e_{i+1} \in S_2^{i+1}. \end{cases} \tag{3.4.5}$$

And $z_{(A,B)} = 0$ for all $(A,B) \neq S^i$ for all $i \in [1,n]$

Qi [44], solved problem (3.4.1) in a different way. He reduces (3.4.1) to a weighted matroid problem

$$\max\{|c|X \; : \; X \in \mathcal{I}(S)\},$$

where $S = \{e_i \; : \; c_i = c(e_i) > 0\}$ and $\mathcal{I}(S)$ as given in (2.1.1).

Lemma 3.4.1 Given $D = (E, \mathcal{I})$ and h its rank function, if

$$\max\{x(X) \; : \; x \in \mathcal{P}_f\} = x^0(X) = |X|, \text{ for } X \in D(E),$$

then $x^0 = \chi_X$ and X is an independent set of the ditroid.

Proof. From (3.3.15) we have that

$$|X| = x^0(X) = h(X). \tag{3.4.6}$$

$$\Rightarrow \; X \in \mathcal{I}, \text{ and, } x^0 = \chi_X. \tag{3.4.7}$$

Hence the lemma holds. □

Corollary 3.4.1 If x is the characteristic vector of some $X \in D(E)$, such that $x(X) > h(X)$, then $X \notin \mathcal{I}$.

3.4.2 g-polymatroids

Hasin [30], considered the following optimization problem associated with a g-polymatroid :

$$\left.\begin{aligned} \max\ cx \\ \text{subject to}\ \ -p(A) \le x(A) \le b(A). \end{aligned}\right\} \tag{3.4.8}$$

The optimization problem associated with the corresponding bisubmodular polyhedron is

$$\left.\begin{aligned} \max\ cx \\ \text{subject to}\ \ x(A,B) \le f(A,B) \end{aligned}\right\} \tag{3.4.9}$$

where $f(A,B) = b(A) + p(B)$, as shown in (3.2.9) and $f(\phi,\phi) = 0 = p(\phi) = b(\phi)$. Hence ggs of (3.4.9) is

$$x_i = x(e_i) = \begin{cases} f(S^i) - f(S^{i-1}) & \text{if } \ c_i > 0 \\ f(S^{i-1}) - f(S^i) & \text{if } \ c_i \le 0. \end{cases} \tag{3.4.10}$$

Now

$$\begin{aligned} f(S^i) - f(S^{i-1}) &= f(S_1^i, S_2^i) - f(S_1^{i-1}, S_2^{i-1}) \\ &= b(S_1^i) + p(S_2^i) - b(S_1^{i-1}) - p(S_2^{i-1}) \\ &= b(S_1^i) - b(S_1^{i-1}). \qquad [\text{since, } e_i \in S_1^i \Rightarrow e_i \notin S_2^i \Rightarrow S_2^i = S_2^{i-1}] \end{aligned}$$

And

$$\begin{aligned} f(S^{i-1}) - f(S^i) &= f(S_1^{i-1}, S_2^{i-1}) - f(S_1^i, S_2^i) \\ &= b(S_1^{i-1}) + p(S_2^{i-1}) - b(S_1^i) - p(S_2^i) \\ &= p(S_2^{i-1}) - p(S_2^i). \qquad [\text{since, } e_i \in S_2^i \Rightarrow e_i \notin S_1^i \Rightarrow S_1^i = S_1^{i-1}]. \end{aligned}$$

So (3.4.10), becomes

$$x_i = x(e_i) = \begin{cases} b(S_1^i) - b(S_1^{i-1}) & \text{if } \ c_i > 0 \\ p(S_2^{i-1}) - p(S_2^i) & \text{if } \ c_i \le 0. \end{cases} \tag{3.4.11}$$

Which is the same as the solution of (3.4.8), given by Hasin [30].

3.4.3 Pseudomatroid Polyhedron

Chandrasekaran and Kabadi [7], consider the following linear program, associated with a pseudomatroid polyhedron.

$$\left.\begin{array}{c} \max\ cx \\ \text{subject to}\ \ x(A,B) \leq b(A,B)\ \ \text{for all}\ (A,B) \in D(E). \end{array}\right\} \tag{3.4.12}$$

Since the function b is a bisubmodular function, therefore the ggs of (3.4.12) is :

$$x_i = x(e_i) = \begin{cases} b(S^i) - b(S^{i-1}) & \text{if } c_i > 0 \\ b(S^{i-1}) - b(S^i) = 0 & \text{if } c_i \leq 0. \end{cases} \tag{3.4.13}$$

Which is the same as the solution given by Chandrasekaran and Kabadi [7].

3.4.4 Degree Sequence Polytope

Consider the following linear program, associated with a degree sequence polytope D_n, defined on the node set V.

$$\left.\begin{array}{c} \max\ \ cx \\ \text{subject to}\ \ x(A,B) \leq f(A,B)\ \ \text{for all}\ (A,B) \in D(V) \\ x_v \geq 0\ \ \text{for all}\ \ v \in V. \end{array}\right\} \tag{3.4.14}$$

Where $f(A,B) = |A|(n-1-|B|)$, $A,B \subseteq V$; $A \cap B = \phi$.

In section 3.2.6, we have shown that this polytope is a bisubmodular polyhedron. Let x be a ggs to 3.3.16, where $|c_1| \geq |c_2| \geq \cdots \geq |c_n|$, and

$$S:\ S^1 \subset S^2 \subset \cdots \subset S^n,\ \ \text{where}\ \overline{S^n} = V \text{ and } S^i = (1,2,\ldots i),$$

is the corresponding chain which generates x. Therefore

$$x_i = \begin{cases} f(S^j) - f(S^{j-1}) & \text{if } c_{i_j} > 0 \\ f(S^{j-1}) - f(S^j) & \text{if } c_{i_j} \leq 0. \end{cases}$$

Now we have

$$f(S^i) = f(S_1^i, S_2^i) = |S_1^i|(n-1-|S_2^i|).$$

So

$$
\begin{aligned}
x_i &= f(S_1^i, S_2^i) - f(S_1^{i-1}, S_2^{i-1}) \qquad \text{if } c_i > 0 \\
&= |S_1^i|(n-1-|S_2^i|) - |S_1^{i-1}|(n-1-|S_2^{i-1}|) \\
&= (n-1-|S_2^i|)(|S_1^i| - |S_1^{i-1} && \text{since } [|S_2^i| = |S_2^{i-1}|] \\
&= (n-1-|S_2^i|), && (3.4.15)
\end{aligned}
$$

or

$$
\begin{aligned}
x_i &= f(S_1^{i-1}, S_2^{i-1}) - f(S_1^i, S_2^i) \qquad \text{if } c_i \le 0 \\
&= |S_1^{i-1}|(n-1-|S_2^{i-1}|) - |S_1^i|(n-1-|S_2^i|) \\
&= |S_1^{i-1}|(n-1-|S_2^{i-1}| - n + 1 + |S_2^i|) && \text{since } [|S_1^{i-1}| = |S_1^i|] \\
&= |S_1^{i-1}|. && (3.4.16)
\end{aligned}
$$

In case $x_i = (n-1-|S_2^i|) = (n-1-|S_2^{i-1}|)$, this implies that node v_i is connected to all the nodes in $V/(S_2^{i-1} + v_i)$.
We show that for all edges (u_i, v_j), $v_j \in V/(S_2^{i-1} + v_i)$, $c_i + c_j \ge 0$.
Given $c_i > 0$ and $v_i \in S_1^i$, $v_j \in V/(S_2^{i-1} + v_i)$, implies that $|c_j| \le |c_i|$ for all $v_j \in V/(S_2^{i-1} + v_i)$.
For all v_j in $V/(S_2^{i-1} + v_i)$ such that $c_j > 0$, $c_i + c_j \ge 0$ and
for all v_j in $V/(S_2^{i-1} + v_i)$ such that $c_j < 0$, since $|c_j| < |c_i|$, follows that $c_i + c_j \ge 0$.

And in case $x_i = |S_1^{i-1}|$, then v_i is connected to all the nodes in S_1^{i-1} and for all $v_j \in S_1^{i-1}$, $c_j \ge |c_i|$, so $c_j + c_i \ge 0$.
Thus the ggs is nothing but the greedy solution obtained in [41]. Peled and Srinivasan [41], point out that even though D_n is not a polymatroid a greedy algorithm solves the corresponding linear programming problem. We have shown here that because D_n is a bisubmodular polyhedron the generalised greedy algorithm works.

3.4.5 Base Polyhedron

In section 3.2.2, we proved that $\mathcal{B}_b = \mathcal{P}_f$, where

$$f(A, B) = b(A) + b(E/B) - b(E),$$

and hence

$$f(A, \phi) = b(A).$$

If x is a ggs of $\mathcal{P}_f$ with respect to c, then

$$x_i = \begin{cases} f(S^i) - f(S^{i-1}) & \text{if } e_i \in S_1^i \\ f(S^{i-1}) - f(S^i) & \text{if } e_i \in S_2^i. \end{cases}$$

Particularly, in this case $S^i = (S_1^i, \phi)\, \forall\, i \in (1, n)$. Hence

$$x_i = b(S^i) - b(S^{i-1}),$$

which is nothing but the greedy solution for the base polyhedron given by Fujishige [26].

3.5 Jump System, 2-SA and Greedy Systems

In this section we first briefly present results in [6] relating jump systems with bisubmodular polyhedron and also some operations on jump systems which help to generate new jump systems. We then consider the $(0, \pm 1)$ extreme point bisubmodular polyhedra and define greedy systems as the collection of $(0, \pm 1)$ vectors of such polyhedra and show that greedy systems satisfy a 2-augmentation property and that ditroids are greedy systems.

3.5.1 2-step axiom Jump System and Bisubmodular Polyhedra

For vectors $x,\ y \in Z^{|E|}$, define the norm of x as $\|x\| = \sum(|x_j|\ :\ e_j \in E)$ and the distance $d(x, y) = \|x - y\|$. For $x,\ y \in Z^{|E|}$, a step from x to y is a vector $s \in Z^{|E|}$ such that $\|s\| = 1$ and $d(x + s, y) = d(x, y) - 1$. Denote the set of steps from x to y by $St(x, y)$. A jump system is a pair $(E, \mathcal{F})$, where $\mathcal{F} \subseteq Z^{|E|}$ satisfies the following 2-step axiom :

(2-SA). If $x,\ y \in \mathcal{F},\ s \in St(x, y)$, and $x + s \notin \mathcal{F}$, then there exists a $t \in St(x + s, y)$ such that $x + s + t \in \mathcal{F}$.

The property $t \in d(x + s, y)$ could be replaced by $t \in d(x, y)$ in the statement of the axiom.

Relationship between jump systems and bisubmodular polyhedra has been shown in [6]. We mention some of these results below for future reference.

Definition 3.5.1 $x \in \mathcal{F}$ is (A, B) maximal in $\mathcal{F}$ if $y \in \mathcal{F}$, $y_j > x_j$ for all $j \in A$, $y_j \leq x_j$ for all $j \in B$ imply $y|_{A \cup B} = x|_{A \cup B}$.

Lemma 3.5.1 If $\mathcal{F}$ satisfies 2-SA and $x, y \in \mathcal{F}$ with $y(A) - y(B) > x(A) - x(B)$, then x is not (A, B)-maximal.

Given $\mathcal{F}$, define f by $f(A, B) = \max_{x \in \mathcal{F}}(x(A) - x(B))$, if the maximum exists, and to be ∞ otherwise.

Lemma 3.5.2 If $\mathcal{F}$ satisfies 2-SA, then f is bisubmodular.

Theorem 3.5.1 If $\mathcal{P}_f$ is a bisubmodular polyhedron and f integral, then $\mathcal{F} = Z^{|E|} \cap \mathcal{P}_f$ satisfies the 2-SA.

Theorem 3.5.2 If $\mathcal{F}$ satisfies 2-SA, then $conv(\mathcal{F})$ is an integral bisubmodular polyhedron.

We now give an alternative way to prove that the degree sequence polytope D_n is a bisubmodular polyhedron.

We first prove the following lemma.

Lemma 3.5.3 Collection of vectors $x \in D_n \cap Z^n$, where D_n is the degree sequence polytope (3.2.17), satisfy 2-SA.

Proof. Let $x, y \in D_n \cap Z^n$. Let $s \in St(x, y)$, then $s = \pm u_i$ for some i. Suppose $s = -u_i$. Then $x + s$ has the degree of the i$^{\text{th}}$ node reduced by '1'.

Since x and y are degree sequences and $s \in St(x, y)$, there must be another node of G say node j, such that $y_j < x_j$.
Choose $t = -u_j$. Then $(x + s + t)$ is also a degree sequence obtained from x, by removing the edge (i, j). The case $s = +u_i$ can be handled in a similar way.

In [41], it has been shown that D_n can be described by the following set of inequalities

$$x(S) - x(T) \leq |S|(n - 1 - |T|),$$

where S, T are the subsets of $\{1, 2, \ldots, n\}$ such that $S \cap T = \phi$. And

$$\max_{x \in D}(x(S) - x(T)) = |S|(n - 1 - |T|) = f(S, T).$$

Therefore by lemma 3.5.2, it follows that $f(S,T)$ is a bisubmodular function and hence D_n is a bisubmodular polyhedron. □

3.5.2 Greedy System

Definition 3.5.2 Define a greedy system as all $\{0, \pm 1\}$ vectors in a bisubmodular polyhedron with $\{0, \pm 1\}$ extreme points.

Some examples of greedy systems are the integer vectors of the pseudomatroid polyhedron and the generalised matroid polyhedron.

Theorem 3.5.3 Let $(E, \mathcal{F})$ be a greedy system. Then **2-augmentation** property holds for $(E, \mathcal{F})$, i.e., if (X_1, X_2) and $(Y_1, Y_2) \in \mathcal{F}$ such that they are non-cancelling and $|(Y_1, Y_2)| \geq |(X_1, X_2)| + 1$, then there exist $e, f \in Y/X$ ($e = f$ allowed) such that $X \cup \{e; f\} \in \mathcal{F}$. Signs of e and f will be same as in Y.

Proof. For any X, $Y \in \mathcal{F}$ that are non-cancelling and $|Y| \geq |X| + 1$, if $X \subseteq Y$, result is obvious by 2-SA. (since, $(E, \mathcal{F})$ is greedy $\Rightarrow$ it satisfies 2-SA.)
If $|X/Y| = 1$ result is again trivial by 2-SA, as follows.

Given $|X/Y| = 1$; and $|Y| \geq |X| + 1$ so $|Y/X| \geq 2$. Let $e_i \in X/Y$.

Now by 2-SA, if $Y + e_i \notin \mathcal{F}$ then $Z = (Y + e_i)/e_j \in \mathcal{F}$ or $Z = (Y + e_i)/e_k \in \mathcal{F}$, for e_j and $e_k \in Y/X$.
For definiteness, let $Z = (Y + e_i)/e_j$.
Now consider X and Z. Since $|Y/X| \geq 2$, so $|Z/X| \geq 1$. Also $X \subseteq Z$. This implies by 2-SA, that there exists $V \subseteq Z/X$ such that $|V| \leq 2$ and $X \cup V \in \mathcal{F}$, but $V \subseteq Y/X$. This proves the result.

When $|X/Y| \geq 2$, let 2-augmentation property not hold in this case. If possible, take a

counter example X, Y with $|X \cap Y|$ maximum amongst all counter examples.

That is for any $e \in Y/X$, $(X + e) \notin \mathcal{F}$, also $(X + e + f) \notin \mathcal{F}$, for all $f \in Y/X$. Since otherwise X and Y will not be a counter example. Thus by 2-SA there exists $g \in X/Y$ such that $(X + e)/g \in \mathcal{F}$. Let $X' = (X + e)/g)$.

Now X' and Y are non-cancelling and $|X' \cap Y| > |X \cap Y|$ and $|X'| \leq |Y| - 1$. Hence by choice of X and Y; X', Y do not form a counter example.

This implies, there exists $\{l, h\} \in Y/X'$ such that $X' \cup \{l, h\} \in \mathcal{F}$. Let $X'' = X' \cup \{l, h\}$, i.e. $X'' = X \cup \{e, l, h\}/g$.

Also X'' and X are non-cancelling and $X'' \geq |X| + 1$ and $|X/X''| = 1$. As shown earlier the result is valid in this case. Hence there exist $\{u, v\} \subseteq \{e, l, h\}$, such that $X \cup \{u, v\} \in \mathcal{F}$. But u and $v \in Y$. This contradicts our assumption. Hence the result. □

Theorem 3.5.4. Any greedy system in which $Y \subseteq X \in \mathcal{I} \Rightarrow Y \in \mathcal{I}$, is a ditroid.

Proof. Let $D = (E,\mathcal{I})$ be a subset system in which $Y \subseteq X \in \mathcal{I} \Rightarrow Y \in \mathcal{I}$ and the greedy algorithm solves the corresponding combinatorial optimization problem. We want to show that $D = (E,\mathcal{I})$ is a ditroid.

Since $Y \subseteq X \in \mathcal{I} \Rightarrow Y \in \mathcal{I}$, so $\phi \in \mathcal{I}$.
It is enough to show that $\mathcal{I}$ satisfies (D3), that is $\mathcal{I}$ satisfies the augmentation property of ditroids.

Let (A, B); $(C, D) \in \mathcal{I}$ with $|(A, B)| = p$ and $|(C, D)| = p + 1$ and non-cancelling. If possible, let there be no $e \in (C, D)/(A, B)$ such that $(A, B) + e \in \mathcal{I}$.
Let us define a weight function, as follows :

$$c(e) = \begin{cases} (p+2) & \text{if } \ e \in A \\ -(p+2) & \text{if } \ e \in B \\ (p+1) & \text{if } \ e \in C/A \\ -(p+1) & \text{if } \ e \in D/B \\ 0 & \text{if } \ e \notin (C, D) \cup (A, B) \end{cases}$$

(A, B) is suboptimal, because $c((C, D)) \geq (p+1)^2 > p(p+2) = c((A, B))$. The greedy algorithm, when applied to this instance, will give (A, B) as the greedy solution, since $(A, B) + e \notin \mathcal{I}$ for all $e \in (C, D)/(A, B)$ and for all other elements of E, the corresponding weight is zero. This contradicts the fact that $(E, \mathcal{I})$ is a greedy system.

Thus $\mathcal{I}$ satisfies the augmentation axiom, (D3). □

Theorem 3.5.5 Every ditroid is a greedy system.

Proof. In section 3.2.7, we have shown that the convex hull of the characteristic vectors of the independent sets of a ditroid is a bisubmodular polyhedron. Thus by definition, a ditroid is a greedy system.

Here we give another proof of theorem (3.5.5) independent of bisubmodularity.

Let $D = (E, \mathcal{I})$ be a ditroid. We will prove that the greedy algorithm works for a ditroid. If not, take a counter example - a ditroid $(E, \mathcal{I})$ for which the greedy algorithm does not work. So there exists a c for which the greedy solution is not optimal. Furthermore it is easy to see that there exists a c for which the greedy solution is unique and non optimal. Choose such a c, and let X^G be the corresponding greedy solution and X^O an optimal solution, such that $|X^G \cap X^O|$ is maximal. We claim that X^G and X^O are non-cancelling and $|X^G| = |X^O|$. First we will show that X^G and X^O are non-cancelling.

If possible let X^G and X^O be cancelling, that is, there exists at least one $e \in E$ such that either $e \in X_1^G$ and $e \in X_2^O$ or $e \in X_2^G$ and $e \in X_1^O$, where $X^G = (X_1^G, X_2^G)$ and $X^O = (X_1^O, X_2^O)$.

Let $e \in X_1^G$ and $e \in X_2^O$. By the greedy algorithm $e \in X_1^G \Rightarrow c(e) > 0$.
Then clearly $c(X^O) < c(X^{O'})$ where $X^{O'} = X^O/e$
$\Rightarrow$ X^O is not an optimal solution.
The case where $e \in X_2^G$ and $e \in X_1^O$, can be handled in a similar way. Hence X^G and X^O are non-cancelling to each other.

We will now prove that $|X^G| = |X^O|$.
If possible let $|X^G| \neq |X^O|$. So either $|X^G| > |X^O|$ or $|X^G| < |X^O|$.

Let $|X^G| > |X^O|$ implies there exists at least one $e \in X^G/X^O$ such that $X^O + e \in \mathcal{I}$ and consequently $cX^O < c(X^O + e)$ [since $c(e) > 0$.]

$\Rightarrow$ X^O is not optimal.

If, $|X^G| < |X^O|$ this would imply that X^G can be augmented and hence is not the greedy solution. So, $|X^G| = |X^O|$ and we have assumed that $|X^G \cap X^O|$ is maximal.

Now, select $e \in X^G \Delta X^O$ such that e is the last element of $X^G \Delta X^O$ as per the order of the gga. That is for any $e_i \in X^G \Delta X^O$, we have $|c(e_i)| > |c(e)|$.

If $e \in X^G/X^O$ then $|X^G/e| = |X^O| - 1$

implies there exists $f \in X^O/X^G$ such that $(X^G/e + f) \in \mathcal{I}$.

Since $|c(e)| < |c(f)|$ this contradicts the fact that X^G is the greedy solution. So e must belongs to X^O/X^G.

But then, $|X^O/e| = |X^G| - 1$ implies there exists $f \in X^G/X^O$, such that $(X^O/e + f) \in \mathcal{I}$. Which contradicts the optimality of X^O. This shows that $X^G = X^O$. □

It will be worth investigating if there are any greedy systems which are neither pseudo-matroids, nor ditroids.

3.5.3 Some More Operations on Greedy System

In section 3.2.8 we considered various operations like restriction and reflection of a bisubmodular polyhedron to obtain other bisubmodular systems. These operations are also valid for greedy systems. In this subsection we give methods to construct greedy systems which are subset systems of a given greedy system.

In [6], Cunningham and Bouchet define a gap of a set $\mathcal{F} \subseteq Z^n$ as an integral point in $\text{conv}(\mathcal{F})/\mathcal{F}$. They then raise the problem that if $\mathcal{F}'$ has been obtained from $\mathcal{F}$ by adding some of the gaps of $\mathcal{F}$, when will $\mathcal{F}'$ satisfy 2-SA. We here tackle the reverse problem that is, what integer points to remove from $\mathcal{F}$ so that it continues to satisfy 2-SA. We manage to give a partial answer, to this problem.

Definition 3.5.3 For $x = (x_1, x_2, \ldots, x_n) \in \mathcal{F}$, we define $\mathcal{F}_1$ in the following way

$$\mathcal{F}_1 = \begin{cases} x & : x \in \mathcal{F} \text{ and } x_1 = x_2 = 0 \\ x - (x_1, 0, 0, \ldots, 0) & : x \in \mathcal{F} \text{ and } x_2 = 0 \text{ but } x_1 \neq 0 \\ x - (0, x_2, 0, \ldots, 0) & : x \in \mathcal{F} \text{ and } x_1 = 0 \text{ but } x_2 \neq 0. \end{cases}$$

Theorem 3.5.6 $\mathcal{F}_1$ satisfies 2-SA.

Proof. Let x' and $y' \in \mathcal{F}_1$ and x, y be the corresponding vectors belonging to $\mathcal{F}$. Note that the choices of x and y are not unique.

It is evident that $St(x', y') \subseteq St(x, y)$. Suppose $s \in St(x', y')$ and $s = \pm u_i$, where u_i is the unit vector in $\Re^n$ whose i$^{\text{th}}$ component is 1, and other components are zero.
Clearly $i > 2$. If $x' + s \notin \mathcal{F}_1$, by definition of $\mathcal{F}_1$, $x + s \notin \mathcal{F}$.
Since $\mathcal{F}$ satisfies 2-SA, there exists a $t \in St(x, y)$ such that $z = (x + s + t) \in \mathcal{F}$.
For all choices of x and y, it is not necessary that $x' + s + t \in \mathcal{F}_1$. That is t may not belong to $St(x', y')$. We show below that the second step can always, be selected, from $St(x', y')$.

case (1) First two components of x as well as y are zero, i.e., $x_1 = x_2 = y_1 = y_2 = 0$.

case (2) First two components of x and y are of the form :

(i) $x = (1, 0, \ldots)$ $y = (1, 0, \ldots)$
(ii) $x = (0, 1, \ldots)$ $y = (0, 1, \ldots)$
(iii) $x = (-1, 0, \ldots)$ $y = (-1, 0, \ldots)$
(iv) $x = (0, -1, \ldots)$ $y = (0, -1, \ldots)$.

In all the above cases $t \in St(x, y) = St(x', y')$ and thus $x' + s + t \in \mathcal{F}_1$.

case (3) First two components of x and y are of the form :

(i) $x = (1, 0, \ldots)$ $y = (-1, 0, \ldots)$
(ii) $x = (-1, 0, \ldots)$ $y = (1, 0, \ldots)$
(iii) $x = (0, 1, \ldots)$ $y = (0, -1, \ldots)$
(iv) $x = (0, -1, \ldots)$ $y = (0, 1, \ldots)$.

In all these four sub cases, it is possible that, $t = \pm u_i$ for $i = 1$ or 2, i.e, t may not belong to $St(x', y')$. But in that case $z = x + s + t$ will have it's first two components equal to zero, and $z' = x + s \in \mathcal{F}_1$, a contradiction. So $t \in St(x', y')$ and hence $z' \in \mathcal{F}_1$.

case (4) First two components of x and y are of the form :

$$\text{(i)} \quad x = (1, 0, \ldots) \quad y = (0, \pm 1, \ldots)$$
$$\text{(ii)} \quad x = (-1, 0, \ldots) \quad y = (0, \pm 1, \ldots).$$

In this case t as in case-3, may not belongs to $St(x', y')$. We will show that, we can always find a step $v = u_j \in St(x', y')$ such that $(x' + s + v) \in \mathcal{F}_1$.
Let $z = (x + s + t) \in \mathcal{F}$ where $t = u_j$ for $j \leq 2$. Consider the vectors z and y, and for $k \leq 2$, choose $m \in St(x, y)$ and $m = \pm u_k$, as the case may be. Such a step exists. Since $\mathcal{F}$ satisfies 2-SA, there exists a $v \in St(x, y)$ such that $w = z + m + v \in \mathcal{F}$, and v must belong to $St(x', y')$. Only one component out of the first two components of w is non-zero, thus $w' \in \mathcal{F}_1$. But $w' = x' + s + v$. The case $t = -u_j$, $j \leq 2$ can be argued in the same way. □

Now by theorem (3.5.2), convex hull of $\mathcal{F}_1$ is a bisubmodular polyhedron, and hence a greedy system.

In case we consider the subset $\overline{\mathcal{F}}_1$ of $\mathcal{F}$ where,

$$\overline{\mathcal{F}}_1 = \{x \in \mathcal{F}, \text{ such that either } x_1 = x_2 = 0 \text{ or } x_1 \neq 0, x_2 = 0 \text{ and } x_1 = 0, x_2 \neq 0\}.$$

Then it follows as a corollary of the above result that $\overline{\mathcal{F}}_1$ is also a greedy system.

Definition 3.5.4 For $x \in \mathcal{F}$, define

$$\mathcal{F}_2 = \{x - (x_1, x_2, 0, \ldots, 0) \; : \; x \in \mathcal{F} \, x_1 \neq 0, \, x_2 \neq 0\}.$$

Theorem 3.5.7 $\mathcal{F}_2$ satisfies 2-SA.

Proof. Let x' and y' belong to $\mathcal{F}_2$ and x, y be the corresponding vectors belonging to $\mathcal{F}$. As noted before, x and y are not unique, but $St(x', y') \subseteq St(x, y)$, for all choices of x and y.

Let $s \in St(x', y')$.
If $x' + s \notin \mathcal{F}_2 \Rightarrow x + s \notin \mathcal{F}$.
There exists a $t \in St(x, y)$, such that $z = (x + s + t) \in \mathcal{F}$. In case the first two components of x and y are equal, $St(x, y) = St(x', y')$ thus $t \in St(x', y')$ and $x' + s + t \in \mathcal{F}_2$.

Consider the case when the first two components of x and y are not equal and $t \notin St(x', y')$. Let $t = +u_1$ or $-u_1$ as the case may be. Let $z = x + s + t$ and consider the vectors z and y. Since $z_1 = 0$, choose $l \in St(z, y)$, $l = +u_1$ or $-u_1$.

If $(z + l) \in \mathcal{F}$, then since $z_1 \neq 0$, $z_2 \neq 0$ $(z + l)' \in \mathcal{F}_2$. But $(z + l)' = (x + s)'$. This gives $(x + s) \in \mathcal{F}$, a contradiction. Hence $z + l \notin \mathcal{F}$. So there exists a $m \in St(x, y)$ such that $w = (z + l + m) \in \mathcal{F}$, and $w' \in \mathcal{F}_2$.

Now if $m = \pm u_j$ for some $j > 2$, $w' \in \mathcal{F}_2$, and $w' = x' + s + m$, where s, $m \in St(x', y')$, implies $\mathcal{F}_2$ satisfies 2-SA.

But if $m = \pm u_2$, ($m = \pm u_1$ not possible) then $w_2 = 0$, so w' does not exists. Choose $v = \mp u_2 \in St(x, y)$, that is if $m = u_2$, then $v = -u_2 \in St(x, y)$ and if $m = -u_2$, then $v = -u_2 \in St(x, y)$. Choose the appropriate v and $v \in St(w, y)$. if $(w + v) \in \mathcal{F}$, then $w' = (w + v)' = (x' + s + m) \in \mathcal{F}_1$ and we are through.

Suppose $(w + v) \notin \mathcal{F}$. Since $v \in St(w, y)$, by 2-SA, there exists a $q \in St(w, y)$ such that $p = (w + v + q) \in \mathcal{F}$. But $q \in St(x', y')$ and p_1, $p_2 \neq 0$. Therefore, $p' = (w + v + q)' = (x' + s + q) \in \mathcal{F}_1$ and s, $q \in St(x', y')$. Hence $\mathcal{F}_2$ satisfies 2-SA. In case $t = u_2$ or $-u_2$, the same arguments can be used by choosing $l = \pm u_2$. □

Again by theorem (3.5.2), convex hull of $\mathcal{F}_2$ is a bisubmodular polyhedron and therefore it is a greedy system.

Let $\mathcal{F}'$ be the restriction of $\mathcal{F}$ to $E/\{e_1, e_2\}$, that is

$$\mathcal{F}' = \{x|_{E/\{e_1,e_2\}} : x \in \mathcal{F}\},$$

then $\mathcal{F}'$ can be decomposed into two greedy systems $\mathcal{F}_1$ and $\mathcal{F}_2$.

Definition 3.5.5 We define $\mathcal{F}_3$ in the following way.

$$\mathcal{F}_3 = \begin{cases} x & : x \in \mathcal{F} \text{ and } x_1 = 0 \ x_2 = 0 \\ x - (x_1, x_2, 0, 0, \ldots, 0) & : x \in \mathcal{F} \text{ and } x_1 \neq 0 \ x_2 \neq 0. \end{cases}$$

Theorem 3.5.8 $\mathcal{F}_3$ satisfies 2-SA.

Proof. Proof will follow from the arguments for $\mathcal{F}_1$ and $\mathcal{F}_2$. □

Here also if we consider $\overline{\mathcal{F}}_3$ as follows :

$$\overline{\mathcal{F}}_3 = \{x \in \mathcal{F} \text{ such that either } x_1 = x_2 = 0 \text{ or } x_1 \neq 0 \text{ and } x_2 \neq 0\},$$

then it can be proved by the above arguments that, $\overline{\mathcal{F}}_3$ satisfies the 2-SA.

Definition 3.5.6 Define $\mathcal{F}_4$ in the following way.

$$\mathcal{F}_4 = \begin{cases} x & : x \in \mathcal{F} \text{ and } x_1 = 0 \; x_2 = 0 \\ x - (x_1, x_2, 0, 0, \ldots, 0) & : x \in \mathcal{F} \text{ and } x_1 \neq 0 \; x_2 \neq 0 \text{ both have the same sign.} \end{cases}$$

Theorem 3.5.9 $\mathcal{F}_4$ satisfies 2-SA.

Proof. Proof will follow from the arguments for $\mathcal{F}_1$ and $\mathcal{F}_2$. □

Definition 3.5.7 Define $\mathcal{F}_5$ in the following way.

$$\mathcal{F}_5 = \begin{cases} x & : x \in \mathcal{F} \text{ and } x_1 = 0 \; x_2 = 0 \\ x - (x_1, x_2, 0, 0, \ldots, 0) & : x \in \mathcal{F} \text{ and } x_1 \neq 0 \; x_2 \neq 0 \text{ have opposite sign.} \end{cases}$$

Theorem 3.5.10 $\mathcal{F}_5$ satisfies 2-SA.

Proof. Again the proof follows from the arguments for $\mathcal{F}_1$ and $\mathcal{F}_2$. □

Chapter 4

POLYHEDRAL STRUCTURE OF A BISUBMODULAR POLYHEDRON

4.1 Introduction

In this chapter we develop the polyhedral structure of the bisubmodular polyhedron and characterize its facets, faces and extreme points. Adjacency criteria for extreme points is also established.

In section two we focus on faces and facets of $\mathcal{P}_f$ and a necessary and sufficient condition for a diset inequality $x(A, B) \leq f(A, B)$ to represent a facet of $\mathcal{P}_f$, is given in terms of non-separability of (A, B) with respect to f. We also show that a facet of $\mathcal{P}_f$ is a bisubmodular polyhedron, a facet of a ditroid polyhedron is again a ditroid polyhedron and a facet of a pseudomatroid polyhedron is also a pseudomatroid polyhedron. Facets of the degree sequence polytope, in our sense, reduce to the facets obtained by Peled and Srinivasan in [41].

In section three, extreme points of $\mathcal{P}_f$ have been defined and a polynomial time algorithm to check if a given vector $x \in \Re^n$ is an extreme point of $\mathcal{P}_f$ is given.

Necessary and sufficient conditions for two extreme points to be adjacent on $\mathcal{P}_f$ have been

discussed in section four. Specialised characterizations of extreme points and adjacency for the ditroid polyhedron and pseudomatroid polyhedron have been obtained.

Finally, in section five, we state and prove the min-max theorem for the bisubmodular polyhedron. We show that the membership problem for $\mathcal{P}_f$ and a bisubmodular function minimization problem are equivalent.

4.2 Faces and Facets

Definition 4.2.1 Let (X_1, X_2) and $(Y_1, Y_2) \in D(E)$ be non-cancelling, then their union $(X_1, X_2) \cup (Y_1, Y_2) = (X_1 \cup Y_1, X_2 \cup Y_2)$, is the least upper bound of (X_1, X_2) and (Y_1, Y_2) and their intersection $(X_1, X_2) \cap (Y_1, Y_2) = (X_1 \cap Y_1, X_2 \cap Y_2)$ is the greatest lower bound. If $\mathcal{D} \subseteq D(E)$ consists of non-cancelling disets and is closed under diset union and intersection, then it forms a **distributive lattice** in $D(E)$.
rank$(\mathcal{D})$ = maximum number of linearly independent characteristic vectors of the disets of $\mathcal{D}$.

We will also call a subset of $\mathcal{D}$ **linearly independent**, if their corresponding characteristic vectors are linearly independent.

Given $x \in \mathcal{P}_f$, we say that a diset $(A, B) \in D(E)$ is x-tight if $x(A, B) = f(A, B)$. Using bisubmodularity of f the following lemma has been proved in [33].

Lemma 4.2.1 If (A, B) and (C, D) are x-tight for some $x \in \mathcal{P}_f$, then the following disets are also x-tight.

1. $(A, B) \cup (C, D) = ((A \cup C)/(B \cup D), (B \cup D)/(A \cup C))$.
2. $(A, B) \cap (C, D) = (A \cap C, B \cap D)$.
3. $(C/B, D/A)$, and
4. $(C \cup (A/D), D \cup (B/C))$.

We borrow the notation from Fujishige [26] and define, for an $x \in \mathcal{P}_f$

$$\mathbf{D}(x) = \{X \in D(E) \,:\, x(X) = f(X)\}. \tag{4.2.1}$$

Lemma 4.2.2 For $x \in \mathcal{P}_f$,

$$\mathbf{D}(x) = \{X \in D(E) \,:\, x(X) = f(X)\}$$

contains a distributive lattice, say $\mathcal{D}$ and $\mathrm{rank}(\mathcal{D}) = \mathrm{rank}(\mathbf{D}(x))$.

Proof. Let (A, B) and $(C, D) \in \mathbf{D}(x)$ be two cancelling but independent disets. Then from lemma 4.2.1, $(C/B, D/A) \in \mathbf{D}(x)$ and $(C/B, D/A)$ is non-cancelling with (A, B). Also (A, B) and $(C/B, D/A)$ are independent, that is their characteristic vectors are linearly independent in $\Re^n$. □

We can therefore collect all non-cancelling disets from $\mathbf{D}(x)$, and these by lemma 4.2.1 will form a distributive lattice $\mathcal{D}$ (say), in which set intersection is the 'meet' operation and set union is the 'join' operation. Also we see from the proof, that $\mathrm{rank}(\mathcal{D}) = \mathrm{rank}(\mathbf{D}(x))$.

As in [26], for a sublattice $\mathcal{D}_0$ of $D(E)$, define

$$F(\mathcal{D}_0) = \{x \in \Re^n,\ \forall\, X \in \mathcal{D}_0,\ x(X) = f(X),\ \forall\, X \in D(E)/\mathcal{D}_0,\ x(X) \leq f(X)\}, \tag{4.2.2}$$

and

$$F^0(\mathcal{D}_0) = \{x \in \Re^n,\ \forall\, X \in \mathcal{D}_0,\ x(X) = f(X),\ \forall\, X \in D(E)/\mathcal{D}_0,\ x(X) < f(X)\}. \tag{4.2.3}$$

Definition 4.2.2 Let $\mathcal{D}_0$ be a sublattice of $D(E)$ and let $F^0(\mathcal{D}_0) \neq \phi$. Then $F(\mathcal{D}_0)$ defines a face of $\mathcal{P}_f$.

If we consider all the disets in $\mathcal{D}_0$, which are non-cancelling, then they themselves form a sublattice of $\mathcal{D}_0$. Thus $\mathcal{D}_0$ may contain more than one sublattice of non-cancelling sets, but each sublattice will have the same rank.

Let $\mathcal{D}_{00}$ be a sublattice of $\mathcal{D}_0$ consisting of all non-cancelling sets of $\mathcal{D}_0$. Suppose $\mathcal{D}_{00}$ consists of disets $A_1, A_2, \ldots, A_m$.

Define $T_1 = A_1$ and T_k as A_k if $A_k \subseteq T_{k-1}$ and T_{k-1} otherwise. Then T_m will be the minimal element of $\mathcal{D}_{00}$, and denote it by S^1. Now find the minimal element of $\mathcal{D}_{00}/S^1$ and denote it by S^2 and so on, till a maximal chain

$$S : \phi \subset S^1 \subset S^2 \subset \cdots \subset S^l,$$

of $\mathcal{D}_{00}$ is constructed. Since $\{S^1, S^2, \ldots, S^l\}$ form a set of maximal linearly independent sets in $\mathcal{D}_{00}$, and hence in $\mathcal{D}_0$, $\text{rank}(\mathcal{D}_0) = l$.

4.2.1 Dimension of a Face

There are several ways of defining the dimensions of a polyhedron $\mathcal{P}_f$. We will use the definition that, if r is the maximum number of linearly independent inequalities satisfied as equalities by all $x \in \mathcal{P}_f$, then dimension $\mathcal{P}_f = n - r$.

If there exists a $x \in \mathcal{P}_f$ such that $x(X) < f(X)$ for all $X \in D(E)$, then $\mathcal{P}_f$ has an interior point and hence it is of full dimension. Let $F(\mathcal{D}_0)$ represent a face of $\mathcal{P}_f$ for some sublattice $\mathcal{D}_0$ of $D(E)$.

Definition 4.2.3 Dimension of a face $F(\mathcal{D}_0)$ of $\mathcal{P}_f$ is equal to $\dim\mathcal{P}_f - k$, where k is the length of a proper maximal chain in $\mathcal{D}_0$.

Definition 4.2.4 If $F(\mathcal{D}_0)$ is a face of dimension $(\dim\mathcal{P}_f - 1)$ of $\mathcal{P}_f$, it is called a **facet** of $\mathcal{P}_f$. In case $\mathcal{P}_f$ is of full dimension, then, $F(\mathcal{D}_0)$ represents a facet if $\dim(F(\mathcal{D}_0)) = n - 1$, that is if and only if $\mathcal{D}_0$ has a maximal chain of length one and $F^0(\mathcal{D}_0) \neq \phi$.

Definition 4.2.5 $F(\mathcal{D}_0)$ will be a facet of $\mathcal{P}_f$ if and only if $\mathcal{D}_0$ consists of a single diset (A, B) such that

$$\begin{aligned} F(\mathcal{D}_0) \;=\; \{x \in \mathcal{P}_f \;:\; & x(A,B) = f(A,B) \text{ and for at least one } x, \text{ tight} \\ & \text{with respect to } (A,B), x(C,D) < f(C,D) \,\forall\, (C,D) \in D(E)\}. \end{aligned}$$

We now introduce the notion of separability of a diset (A, B) with respect to $\mathcal{P}_f$ and show that (A, B) represents a facet of $\mathcal{P}_f$, if and only if (A, B) is non-separable in $\mathcal{P}_f$.

Definition 4.2.6 $(A,B) \in D(E)$ is said to be separable with respect to f if there exists $(X_1,X_2),(Y_1,Y_2) \in D(E)$, such that

$$(X_1,X_2),(Y_1,Y_2) \neq \phi,\ (X_1,X_2) \cap (Y_1,Y_2) = \phi \text{ and } (X_1,X_2) \cup (Y_1,Y_2) = (A,B)$$

$$\text{and } f(X_1,X_2) + f(Y_1,Y_2) = f(A,B). \tag{4.2.4}$$

Consider the case when $f(A,B) = b(A) + p(B)$, where b and p are submodular functions on E, and $\mathcal{P}_f = Q(p,b)$ is the generalised polymatroid.

If for some $Y \subseteq E$, there exists $X \subseteq Y$ such that

$$b(Y) = b(X) + b(Y/X).$$

This implies that

$$f(Y,\phi) = f(X,\phi) + f(Y/X,\phi),$$

and hence Y is separable with respect to f. Frank and Tardos [24] call X, the inner separator of Y with respect to b.

Similarly if there exists $Z \subseteq E$ such that

$$b(Y \cup Z) + p(Z) = b(Y).$$

This implies that

$$f(Y \cup Z,\phi) + f(\phi,Z) = f(Y,\phi),$$

and Y is again separable with respect to f. Frank and Tardos [24] call Z, the outer separator of Y with respect to b.

Inner and outer separabliy of $Y \subseteq E$ with respect to p will also be derivable from (4.2.4).

For a ditroid polyhedron $\mathcal{P}_h$, independent disets will be separable. Separators of a set with respect to the submodular polyhedron and matroid polyhedron can also be derived from (4.2.4). In the theorem below we extend the results in [26] to the bisubmodular case.

Theorem 4.2.1 $(C,D) \in D(E)$ defines a facet of $\mathcal{P}_f$ if and only if, (C,D) is non-separable.

Proof Let the rows of A denote the characteristic vectors of $D(E)$ minus the characteristic vectors of (C,D) and some $(F,G)\ (\neq (C,D))$ and let f, denote the diset function from $D(E)$ to $\Re$, and $f_1 = f(C,D), f_2 = f(F,G), a = \chi_{(C,D)}, b = \chi_{(F,G)}$.
(C,D) does not represent a facet of $\mathcal{P}_f$ if and only if, the system,

$$\begin{aligned} Ax &\leq f \\ ax &= f_1 \\ bx &< f_2 \end{aligned}$$

has no solution.

That is if and only if the system,

$$\left.\begin{aligned} -Ax + fu &\geq 0 \\ ax - f_1 u &= 0 \\ -bx + f_2 u &> 0 \\ u &> 0 \end{aligned}\right\} \tag{4.2.5}$$

has no solution.

By Motzkin's theorem it implies that (C,D) does not represent a facet if and only if the system

$$\begin{aligned} -A^T y + a^T z - b^T t &= 0 \\ f^T y - f_1 z + f_2 t + l &= 0 \\ t,\ l \geq 0,\ l + t > 0,\ y &\geq 0 \end{aligned}$$

has a solution.

In case $t > 0$, this implies that

$$\left.\begin{aligned} -A^T y + a^T z &= b^T \\ f^T y - f_1 z &\leq -f_2 \\ y &\geq 0 \end{aligned}\right\} \tag{4.2.6}$$

has a solution, and putting $t = 0,\ l > 0$ gives

$$\left.\begin{aligned} -A^T y + a^T z &= 0 \\ f^T y - f_1 z &> 0 \\ y &\geq 0 \end{aligned}\right\} \tag{4.2.7}$$

has a solution.

In order for (4.2.6) to have a solution, $z \neq 0$, since b^T is not a row of A^T.

Let $z > 0$. Since A, a and b are all $0, \pm 1$ this implies $z = 1$ and $y_i = 1$ for some i (all other $y_j = 0$) such that $-a_i + a = -b$,

that is, $\quad a = a_i - b$, where $a_i = \chi_{(X,Y)}$.

or, $\quad (C, D) = (X, Y) \cup (F, G)$ and $(X, Y) \cap (F, G) = \phi$.

From the second inequality, we have

$$\begin{aligned} f_1 &\geq f_i + f_2 \\ \text{or} \quad f(C, D) &\geq f(X, Y) + f(F, G). \end{aligned}$$

From submodularity of f, it follows that

$$f(C, D) = f(X, Y) + f(F, G).$$

Therefore, (C, D) represents a facet if and only if, it is not separable.

For $Z < 0$, (4.2.6) does not have a solution. Also (4.2.7) does not have a solution. □

We demonstrate the above results through the following example.

Example 4.2.1 Consider the polyhedron, $\mathcal{P}$, on $D(E)$, where $E = \{e_1, e_2\}$

$$\begin{aligned} x_1 &\leq 1 \\ -x_1 &\leq 1 \\ x_2 &\leq 1 \\ -x_2 &\leq 1 \\ x_1 - x_2 &\leq 1 \\ -x_1 + x_2 &\leq 0 \\ x_1 + x_2 &\leq 2 \\ -x_1 - x_2 &\leq 2 \end{aligned}$$

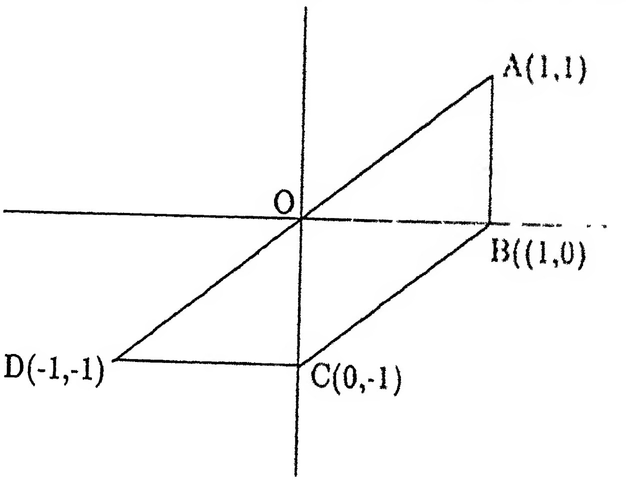

Fig. 4.2.1

defining

$$f(A, B) = \max_{x \in \mathcal{P}} x(A, B),$$

gives $f(e_1,\phi) = 1$, $f(\phi,e_1) = 1$, $f(e_2,\phi) = 1$, $f(\phi,e_2) = 1$, $f(e_1,e_2) = 1$, $f(e_2,e_1) = 0$, $f(e_1e_2,\phi) = 2$ and $f(\phi,e_1e_2) = 2$.

Since $\mathcal{P} \cap Z^2$ satisfies 2-step axiom, f is a bisubmodular function on $D(E)$. $x_1 = 1$, defines a facet of $\mathcal{P}$, we now show that (e_1,ϕ) is non-separable.

By definition (e_1,ϕ) does not have a non-trivial subset, and (ϕ,e_2) is the only possible candidate for being a separator. But

$$\begin{aligned} f((e_1,\phi)\cup(\phi,e_2)) + f(e_2,\phi) &= f(e_1,e_2) + f(e_2,\phi) \\ &= 1+1 \neq f(e_1,\phi). \end{aligned}$$

It can be easily verified that $-x_2 = 1$, $x_1 - x_2 = 1$ and $-x_1 + x_2 = 0$ represent the other three facets, and the corresponding disets are non-separable.

$-x_1 = 1$ does not represent a facet, since $(\phi,e_1) = (e_2,e_1)\cup(\phi,e_2)$ and

$$f(e_2,e_1) + f(\phi,e_2) = 0 + 1 = f(\phi,e_1).$$

Similarly it can be shown that (e_2,ϕ), $(e_1\,e_2,\phi)$ and $(\phi,e_1\,e_2)$ do not represent facets of $\mathcal{P}$, since they are separable.

In [41] Peled and Srinivasan have given necessary and sufficient conditions for (S,T), where S, $T \subseteq V$ and $S \cap T = \phi$, to define a facet of a degree sequence polytope D_n.

We show here that all (S,T) defining facets of D_n are non-separable.

Consider the case when, $n \geq 3$, $|S \cup T| > 1$ and $|S| = |T| \neq \phi$.

(S,T) does not have a separator such that $(A,B) \subseteq (S,T)$ and $f(A,B)+f(S/A,T/B) = f(S,T)$.

Let $(A,B) \subseteq (S,T)$, $(A,B) \in D(V)$ and let $|S| = s$, $|T| = t$, $|A| = a$ and $|B| = b$. Then

$$\begin{aligned} f(A,B) + f(S/A,T/B) &= a(n-1-b) + (s-a)(n-1-t+b) \\ &= s(n-1-t) + a(t-b) + b(s-a). \end{aligned} \quad (4.2.8)$$

Since $a(t-b)$ and $b(s-a)$ both cannot be zero simultaneously, it follows that

$$f(A,B) + f(S/A,T/B) > f(S,T) \text{ for all } (A,B) \subseteq (S,T).$$

For $(A,B) \in D(V)$, consider

$$\begin{aligned} f((S,T)\cup(A,B)) + f(B,A) &= (s+a)(n-1-t-b) + b(n-1-a) \\ &= s(n-1-t) + a(n-1-t-b) + \\ &\quad\; b(n-1-a-s). \end{aligned} \tag{4.2.9}$$

In case $|S \cup T| = n$, to find (X_1,X_2), $(Y_1,Y_2) \not\subseteq (S,T)$ such that $(X_1,X_2)\cap(Y_1,Y_2) = \phi$ and $(X_1,X_2)\cup(Y_1,Y_2) = (S,T)$ not possible. Thus (S,T) in this case has no separators.

For $|S \cup T| = (n-1),(n-2)$, we can select a,b such that $n-1-t-b = 0$ and $n-1-a-s = 0$ and hence,

$$f((S,T)\cup(A,B)) + f(B,A) = f(S,T),$$

that is for $|S\cup T| = (n-1),(n-2)$, (S,T) does not define a facet of D_n.

For $|S\cup T| = 2,3,\ldots,(n-3)$, it can be easily seen that

$$a(n-1-t-b) + b(n-1-a-s) > 0.$$

Thus for all (S,T) such that $|S\cup T| = 2,3,\ldots,(n-3),n$, (S,T) defines a facet of D_n.

All these results tally with the results in [41].

In case $(S,T) = (\phi,i)$, $i\in V$, no subset of (ϕ,i) can be a separator and for the separators $(A,B+i)$ and (B,A), (4.2.9) reduces to :

$$\begin{aligned} f((\phi,i)\cup(A,B)) + f(B,A) &= a(n-1-1-b) + b(n-1-a) \\ &= a(n-2-b) + b(n-1-a) > 0 = f(\phi,i). \end{aligned}$$

That is (ϕ,i) has no separator for $n\geq 4$. This verifies the result in [41] that $x_i = 0$ or $-x_i = 0$ is a facet of D_n, for $n\geq 4$.

Consider the case, when $(S,T) = (i,\phi)$, $i\in V$.
The only separator if possible, are of the type (B,A) and $(A+i,B)$ and in this case (4.2.9) reduces to :

$$\begin{aligned} f((i,\phi)\cup(A,B)) + f(B,A) &= (n-1) + a(n-1-b) + b(n-1-a-1) \\ &= (n-1) + a(n-1-b) + b(n-2-a) > n-1. \end{aligned}$$

Therefore (i, ϕ) has no separators for all $n \geq 4$, and for all possible choices of a, b, such that $ab \neq 0$. $x_i = (n-1)$ has been shown to define a facet of D_n, in [41].

For a fixed $T \in \mathcal{D}$, define

$$F(T) = \{\, x \in \mathcal{P}_f \,:\, x(T) = f(T)\}.$$

To show that a face of a bisubmodular polyhedron is also a bisubmodular polyhedron, we prove the following result.

Theorem 4.2.2 $F(T)$ is the bisubmodular polyhedron.

$$\mathcal{P}_{f_T} = \{x \in \mathcal{P}_f \,:\, x(A) \leq f_T(A);\ \forall\, A \in D(E)\},$$

where

$$f_T(A) = f(A \cup T) + f(A \cap T) - f(T).$$

Proof. Let $x \in F(T)$, then

$$\begin{aligned} x(X) &= x(X \cap T) + x(X \cup T) - x(T) \\ &\leq f(X \cap T) + f(X \cup T) - f(T) = f_T(X) \\ &\Rightarrow x \in \mathcal{P}_{f_T}. \end{aligned}$$

Now, let $x \in \mathcal{P}_{f_T}$, then

$$\begin{aligned} & f(T) = f_T(T) \geq x(T) = -x(-T) \geq -f_T(-T) = f(T) \\ \Rightarrow\ & x(T) = f(T) \\ \Rightarrow\ & x \in F(T) \\ \Rightarrow\ & F(T) = \mathcal{P}_{f_T}. \end{aligned}$$

To show that $\mathcal{P}_{f_T}$ is a bisubmodular polyhedron, it will suffice to prove that, f_T is a bisubmodular function.

Let $A,\ B \in D(E)$, then

$$f_T(A) + f_T(B) \;=\; f(A \cup T) + f(A \cap T) + f(B \cup T) + f(B \cap T) - 2f(T)$$

$$
\begin{aligned}
&\geq f((A \cup T) \cup (B \cup T)) + f((A \cup T) \cap (B \cup T)) + \\
&\qquad f((A \cap T) \cup (B \cap T)) + f((A \cap T) \cap (B \cap T)) - 2f(T) \\
&\geq f((A \cup B) \cup T) + f((A \cap B) \cap T) - 2f(T) + \\
&\qquad f(((A \cup T) \cap (B \cup T)) \cup ((A \cap T) \cup (B \cap T))) + \\
&\qquad\quad f(((A \cup T) \cap (B \cup T)) \cap ((A \cap T) \cup (B \cap T))) \\
&= f((A \cup B) \cup T) + f((A \cap B) \cap T) - 2f(T) + \\
&\qquad f((A \cup B) \cap T) + f((A \cap B) \cup T) \\
&= \{ f((A \cup B) \cup T) + f((A \cup B) \cap T) - f(T) \} + \\
&\qquad \{ f((A \cap B) \cup T) + f((A \cap B) \cap T) - f(T) \\
&= f_T(A \cup B) + f_T(A \cap B).
\end{aligned}
$$

Hence, f_T is a bisubmodular function. This shows that $F(T)$ is a bisubmodular polyhedron. $\square$

4.2.2 Facets of Other Bisubmodular Polyhedra

Consider the ditroid polyhedron $\mathcal{P}_h$, and let T be non-separable with respect to $\mathcal{P}_h$. Then

$$h_T(A) = h(A \cup T) + h(A \cap T) - h(T)$$

can be shown to be a ditroid rank function, and hence the lemma below follows.

Lemma 4.2.3 Every facet of a ditroid polyhedron is also a ditroid polyhedron.

Proof. Let $T \in D(E)$ define a facet of a ditroid $D = (E, \mathcal{I})$, and let h denote its rank function. We will show that h_T is the rank function, of a ditroid.

By definition of h_T we have,

$$
\begin{aligned}
h_T(\phi) &= h(\phi \cup T) + h(\phi \cap T) - h(T) \\
&= h(T) - h(T) = 0.
\end{aligned}
$$

Let $A \in D(E)$, then

$$h_T(A) = h(A \cup T) + h(A \cap T) - h(T)$$

$$\leq \quad h(A) + h(T) - h(T) = h(A) \leq |A|,$$

since h is a rank function of $D = (E,\mathcal{I})$.

Let $A \subseteq B$ then,

$$\begin{aligned} h_T(A) &= h(A \cup T) + h(A \cap T) - h(T) \\ &\leq h(B \cup T) + h(B \cap T) - h(T) = h_T(B), \qquad \text{since } h \text{ is monotonic.} \end{aligned}$$

For $A,\ B \in D(E)$,

$$\begin{aligned} h_T(A) + h_T(B) &= h(A \cup T) + h(A \cap T) + h(B \cup T) + h(B \cap T) - 2h(T) \\ &\geq h((A \cup T) \cup (B \cup T)) + h((A \cup T) \cap (B \cup T)) + \\ &\qquad h((A \cap T) \cup (B \cap T)) + h((A \cap T) \cap (B \cap T)) - 2h(T) \\ &\geq h((A \cup B) \cup T) + h((A \cap B) \cap T) - 2h(T) + \\ &\qquad h(((A \cup T) \cap (B \cup T)) \cup ((A \cap T) \cup (B \cap T))) + \\ &\qquad\quad h(((A \cup T) \cap (B \cup T)) \cap ((A \cap T) \cup (B \cap T))) \\ &= h((A \cup B) \cup T) + h((A \cap B) \cap T) - 2h(T) + \\ &\qquad h((A \cup B) \cap T) + h((A \cap B) \cup T) \\ &= \{\, h((A \cup B) \cup T) + h((A \cup B) \cap T) - h(T) \,\} + \\ &\qquad \{\, h((A \cap B) \cup T) + h((A \cap B) \cap T) - h(T) \\ &= h_T(A \cup B) + h_T(A \cap B). \end{aligned}$$

Hence h_T is a bisubmodular function. Also h_T is integer valued, since h is. This shows that h_T is a ditroid rank function. Thus $\mathcal{P}_{h_T}$ is a ditroid polyhedron. □

As in lemma 4.2.3, let $T \in D(E)$ represent a facet of $\mathcal{P}_h$. Then

$$h_T(A) = h(A) \text{ for all } A \subseteq T.$$

Let D_T denote the ditroid represented by $\mathcal{P}_{h_T}$. We see that D_T is the restriction of the ditroid $D = (E,\mathcal{I})$ to T.

It has already been shown in [24] that every facet of a g-polymatroid is again a g-polymatroid.

(i) $e_i \in T_1$ (ii) $e_i \in T_2$ (iii) $e_i \notin T$

case (i). $e_i \in T_1$.

Then, $b_T(e_i, \phi) = b(e_i, \phi) + b(T) - b(T) = b(e_i, \phi) = 1$ or 0.

case (ii). $e_i \in T_2$. Then

$$\begin{aligned} b_T(e_i, \phi) &= b(T \cap (e_i, \phi)) + b(T \cup (e_i, \phi)) - b(T) \\ &= b(\phi, \phi) + b((T_1, T_2) \cup (e_i, \phi)) - b(T_1, T_2) \\ &= b(T_1, T_2') - b(T_1, T_2), \qquad \text{where } T_2' = T_2/e_i \\ &\geq 0. \qquad\qquad [\text{ by third property of } b.] \end{aligned}$$

Again

$$\begin{aligned} & b(T_1, T_2') \leq b(T_1, T_2) + b(e_i, \phi) \\ \Rightarrow\ & b(T_1, T_2') - b(T_1, T_2) \leq b(e_i, \phi) \\ \Rightarrow\ & b(T_1, T_2') - b(T_1, T_2) \leq 0 \text{ or } 1. \end{aligned}$$

This implies that $b_T(e_i, \phi) = 0$ or 1.

case (iii). $e_i \notin T$, then

$$\begin{aligned} b_T(e_i, \phi) &= b(T \cap (e_i, \phi)) + b(T \cup (e_i, \phi)) - b(T) \\ &= b((T_1, T_2) \cup (e_i, \phi)) + b(\phi, \phi) - b(T_1, T_2) \\ &= b(T_1', T_2) - b(T_1, T_2) \qquad \text{where } T_1' = T_1 + e_i \\ &\geq 0. \qquad\qquad [\text{ by third property of } b.] \end{aligned}$$

Again

$$\begin{aligned} & b(T_1', T_2) \leq b(T_1, T_2) + b(e_i, \phi) \\ \Rightarrow\ & b(T_1', T_2) - b(T_1, T_2) \leq b(e_i, \phi) \\ \Rightarrow\ & b(T_1', T_2) - b(T_1, T_2) \leq 0 \text{ or } 1. \end{aligned}$$

This shows that $b_T(e_i, \phi) = 0$ or 1. Hence in all cases $b_T(e_i, \phi) = 0$ or 1.

Property (3) Let $A_1 \subseteq B_1$, $B_2 \subseteq A_2$ for (A_1, A_2), $(B_1, B_2) \in D(E)$. We show that

$$b_T(A_1, A_2) \leq b_T(B_1, B_2).$$

To prove the above, we will first show that if $A_1 \subseteq B_1$ $B_2 \subseteq A_2$ and (A_1, A_2), (B_1, B_2) belong to $D(E)$ and $T = (T_1, T_2)$ then,

(i) $A_1 \cap T_1 \subseteq B_1 \cap T_1$.

(ii) $B_2 \cap T_2 \subseteq A_2 \cap T_2$.

(iii) $(A_1 \cup T_1)/(A_2 \cup T_2) \subseteq (B_1 \cup T_1)/(B_2 \cup T_2)$.

(iv) $(B_2 \cup T_2)/(B_1 \cup T_1) \subseteq (A_2 \cup T_2)/(A_1 \cup T_1)$.

(i) and (ii) are easy, since $A_1 \subseteq B_1 \Rightarrow A_1 \cap T_1 \subseteq B_1 \cap T_1$
and $B_2 \subseteq A_2 \Rightarrow B_2 \cap T_2 \subseteq A_2 \cap T_2$.
(iii) Let $e \in (A_1 \cup T_1)/(A_2 \cup T_2)$
$\Rightarrow e \in (A_1 \cup T_1)$ and $e \notin (A_2 \cup T_2)$
$\Rightarrow e \in A_1$ or T_1 and $e \notin A_2$ and $e \notin T_2$
$\Rightarrow e \in B_1$ or T_1 and $e \notin B_2$ and $e \notin T_2$
$\Rightarrow e \in (B_1 \cup T_1)/(B_2 \cup T_2)$.
Thus $(A_1 \cup T_1)/(A_2 \cup T_2) \subseteq (B_1 \cup T_1)/(B_2 \cup T_2)$.
(iv) Let $e \in (B_2 \cup T_2)/(B_1 \cup T_1)$
$\Rightarrow e \in (B_2 \cup T_2)$ and $e \notin (B_1 \cup T_1)$
$\Rightarrow e \in B_2$ or T_2 and $e \notin B_1$ and $e \notin T_1$
$\Rightarrow e \in A_2$ or T_2 and $e \notin A_1$ and $e \notin T_1$
$\Rightarrow e \in (A_2 \cup T_2)/(A_1 \cup T_1)$.
Thus $(B_2 \cup T_2)/(B_1 \cup T_1) \subseteq (A_2 \cup T_2)/(A_1 \cup T_1)$.
Now, using the bisubmodularity of b and the above four relations, we have

$$\begin{aligned}
b_T(A_1, A_2) &= b((A_1, A_2) \cap (T_1, T_2)) + b((A_1, A_2) \cup (T_1, T_2)) - b(T) \\
&= b(A_1 \cap T_1, A_2 \cap T_2) + b(A_1 \cup T_1/A_2 \cup T_2, A_2 \cup T_2/A_1 \cup T_1) - b(T) \\
&\leq b(B_1 \cap T_1, B_2 \cap T_2) + b(A_1 \cup T_1/A_2 \cup T_2, A_2 \cup T_2/A_1 \cup T_1) - b(T) \\
&= b((B_1, B_2) \cap (T_1, T_2)) + b((B_1, B_2) \cup (T_1, T_2)) - b(T) \\
&= b_T(B_1, B_2).
\end{aligned}$$

Hence $\mathcal{P}_{b_T}$ or $F(T)$ is a pseudomatroid polyhedron, with b_T as the defining rank function. □

4.3 Extreme Points

In this section we characterize extreme points of $\mathcal{P}_f$. It is shown that extreme points on the submodular polyhedron, the pseudomatroid polyhedron, the degree sequence polyhedron etc. are all obtainable as special cases of the these results. For ease of presentation we will throughout assume that dim $\mathcal{P}_f = n$.

Definition 4.3.1 $x \in \mathcal{P}_f$ is an **extreme point** of $\mathcal{P}_f$, if it is a zero dimensional face of $\mathcal{P}_f$. That is if rank $F(\mathcal{D}) = n$, where $\mathcal{D}$ is a set of all non-cancelling x-tight sets and $F(\mathcal{D})$ is as defined in (4.2.3).

From the definition of the dimension of a face, it follows that if $x \in \mathcal{P}_f$ is an extreme point, then there exists at least one maximal chain of x-tight sets of length n and let the chain be denoted by $S : \phi = S^0 \subset S^1 \subset \cdots \subset S^n$ in D, where $\overline{S^n} = E$. This implies that the i$^{\text{th}}$ component of x is

$$x_i = x(e_i) = \begin{cases} f(S^i) - f(S^{i-1}) & \text{if } \ e_i \in S_1^i \\ f(S^{i-1}) - f(S^i) & \text{if } \ e_i \in S_2^i \end{cases} \tag{4.3.1}$$

for each $i = 1, 2, \ldots, n$.

Thus, if $\mathcal{D}$ is a collection of all the non-cancelling x-tight disets for x an extreme point of $\mathcal{P}_f$, then $\mathcal{D}$ is a **simple** distributive lattice. It is to be noted that, there may be more than one set of non-cancelling tight disets, and hence more than one simple lattice of tight disets corresponding to the same extreme point.

Theorem 4.3.1 Let $\mathcal{P}_f$ be a fully dimensional bisubmodular polyhedron. $x \in \mathcal{P}_f$ is an extreme point of $\mathcal{P}_f$ if and only if there exists a complete chain $S : \phi = S^0 \subset S^1 \subset \cdots \subset S^n$ of x-tight disets in $D(E)$.

Referring back to the example 4.2.1, consider the extreme point $A = (1, 1)$. The tight sets corresponding to this point are $x_1 = 0$, $x_2 = 1$, $-x_1 + x_2 = 0$ and $x_1 + x_2 = 2$.

The two sets of non-cancelling tight sets are $x_1 = 1$, $x_2 = 1$, $x_1 + x_2 = 2$ and $x_2 =$

, $-x_1 + x_2 = 0$, and the two chains corresponding to the first set of tight constraints are :

$$S^1 = (e_1, \phi),\ S^2 = (e_1\, e_2, \phi) \text{ and } T^1 = (e_2, \phi),\ T^2 = (e_2\, e_1, \phi),$$

nd the chain corresponding to the second set of tight constraints is :

$$Z^1 = (e_2, \phi) \text{ and } Z^2 = (e_2, e_1).$$

f only the facet inequalities were considered in describing $\mathcal{P}$, then the second set of the tight onstraints will be the only tight constraint with respect to $A = (1, 1)$ and hence only one hain Z, of tight sets.

If $\mathcal{P}_f$ is a submodular polyhedron, definition 4.3.1 reduces to the definition of an extreme oint given by Fujishige [25].

lternative definition of an extreme point of $\mathcal{P}_f$.

We can also obtain extreme points of $\mathcal{P}_f$ by using the generalised greedy algorithm given n section (3.3).

or a cost vector $c \in \Re^n$, consider the linear programming problem

$$\max\ cx$$
$$\text{subject to } x \in \mathcal{P}_f.$$

efine $P(c) = \{\, i_1, i_2 \ldots, i_n \mid |c_{i_1}| \geq |c_{i_2}| \geq \cdots \geq |c_n| \,\}$ as before.

Let $p = (i_1,\ i_2 \ldots, i_n) \in P(c)$, and we order the elements of E according to p. If x is he ggs for the above linear program, we see from (3.3.3) that it is a face of dimension zero nd hence an extreme point of $\mathcal{P}_f$. In fact x is the ggs corresponding to all permutations $\in P(c)$. For a given c, we notice that it is the ordering of the values of c_i's, rather than heir actual values which define the ggs. Thus all extreme points of $\mathcal{P}_f$ will be obtainable, s the ggs corresponding to the $n!$ permutations of $\{1,\ 2, \ldots, n\}$. Therefore $n!$ is an upper ound on the number of extreme points of $\mathcal{P}_f$. In case f is a polymatroid function, the efinition reduces to the one given in [54].

Given a point $x \in \Re^n$, we want to know, whether the point x is an extreme point of $\mathcal{P}_f$ or not. The following algorithm will answer this question.

Algorithm

MAIN PROGRAM

begin

$N = \{1,2,3,\ldots\ldots n\}$

Initialization $\quad N_1 = N$ and $N_2 = N$

$k = 0,\ A^k = \phi,\ B^k = \phi$

while $\quad N_2 \neq \phi$

begin

choose $\quad i \in N_2$

call subroutine (A^k, B^k, i, N_1, N_2,).

If (F_3) or (F_4), then $\quad N_2 = N_2 - \{i\}$

end

If $N_1 = N_2 = \phi$, then x is an extreme point, if $N_2 = \phi$ but $N_1 \neq \phi$, then x is not a extreme point of $\mathcal{P}_f$.

end

Subroutine (A^k, B^k, i, N_1, N_2,).

begin

$T_1 = (A^k + e_i, B^k)x = f(A^k + e_i, B^k)$

$T_2 = (A^k, B^k + e_i)x = f(A^k, B^k + e_i)$

$F_1 = (T_1)$ .AND. (NOT T_2)

$F_2 =$ (NOT T_1) .AND. (T_2)

$F_3 = (T_1)$ .AND. (T_2)

$F_4 =$ (NOT T_1) .AND. (NOT T_2)

If (F_1)

begin

Then $k = k + 1,\ \hat{e}_k = e_i$

$A^k = A^{k-1} + \hat{e}_k,\ B^k = B^{k-1}$

$N_1 = N_1/\{i\},\ N_2 = N_1$

end

If (F_2)

begin

Then $k = k+1$, $\bar{r}_k = r$,

$A^k = A^{k-1}$, $B^k = B^{k-1} + \bar{r}_k$

$N_1 = N_1/\{i\}$, $N_2 \quad N_1$

end

end

Validity of the Algorithm.

$\underline{N_1 = \phi}$. In this case a chain of length n, consisting of tight sets with respect to x has been found and hence x is an extreme point of $\mathcal{P}_f$.

$\underline{N_1 \neq \phi}$. Let $|N_1| = n - r > 0$. In this case the algorithm stops after finding a chain $S^1 \subset S^2 \subset \cdots \subset S^r$, of length r of tight disets with respect to x. We will now show that if x is an extreme point than the algorithm would not have stopped at S^r.
Let T be a chain of length n of tight disets with respect to x.

case (i) T is non-cancelling with S^r. Either $T^1 \cap S^1 = \phi$, in which case $S^r \cup T^1$ is tight with respect to x and the algorithm would not have stopped at S^r. Or T^j is the first set of T, such that $T^{j-1} \subseteq S^r$ but $T^j \not\subseteq S^r$. In this case $S^r \cup (T^j/T^{j-1})$ is again x tight, and the algorithm could not have stopped at S^r.

case (ii). T is cancelling with S^r. Let j be the first index for which $\overline{T^j}/\overline{S^r} \neq \phi$. Let $Z = T^j/S^r$, then $|Z| = 1$.
Also $(T_1^j/S_2^r, T_2^j/S_1^r)$ is x-tight and hence $(S_1^r \cup (T_1^j/S_2^r), S_2^r \cup (T_2^j/S_1^r))$ is also x-tight. But $(S_1^r \cup (T_1^j/S_2^r), S_2^r \cup (T_2^j/S_1^r)) = S^r \cup Z$.

Thus again the algorithm could not have stopped at S^r. Therefore x is not an extreme point of $\mathcal{P}_f$.

It may be possible to generate efficiently all the chains with respect to an extreme point

$x \in \mathcal{P}_f$, provided it can be shown that for a full dimensional polyhedron, the number of chains will be bound by a polynomial in the dimension of $\mathcal{P}_f$.

In case f is also non-decreasing, then f is a rank function (since we have throughout assumed that $f(\phi, \phi) = 0$.) and the above algorithm will simplify to a great extent, by defining $N_1 = N_2 = \{i \in N : x_i \neq 0\}$. Then for all $i \in N_1 = N_2$, F_3 will not occur.

4.3.1 Extreme Points of Different Kinds of Bisubmodular Polyhedra

(1) Ditroid Polyhedron.

Consider the ditroid $D = (E, \mathcal{I})$. Qi, L. ([43], [44]), defines a ditroid polyhedron as

$$\mathcal{P}_h = \{x \in \Re^n \ : \ x(A) \leq h(A) \text{ for all } A \in D(E)\}, \tag{4.3.2}$$

where h is a ditroid rank function. h defined on $D(E)$ is bisubmodular. Hence $\mathcal{P}_h$ is a bisubmodular polyhedron. If $x \in \mathcal{P}_f$ is its extreme point, then x is generated by some chain

$$S : \phi = S^0 \subset S^1 \subset \cdots \subset S^n \qquad \text{where } S^i = \{e_1, e_2, \cdots, e_i\} \text{ and } \overline{S^n} = E,$$

and

$$x_i = x(e_i) = \begin{cases} h(S^i) - h(S^{i-1}) & \text{if } e_i \in S_1^i \\ h(S^{i-1}) - h(S^i) & \text{if } e_i \in S_2^i \end{cases}$$

where the elements have been numbered with respect to the chain S.

Since h is the rank function, it follows that x is the characteristic vector of a diset $(A, B) \subseteq (S_1^n, S_2^n)$. Also

$$x(A, B) = |A \cup B| = h(A, B).$$

This implies that (A, B) is an independent diset, and $\mathcal{P}_h$ is the convex hull of the characteristic vectors of its independent disets [43].

(2) Base Polyhedron.

Fujishige, S. ([26], [28]), defines a base polyhedron as

$$B_b = \{x \in \Re^n \ : \ x(E) = b(E), \ x(A) \leq b(A) \text{ for all } A \subseteq E\},$$

where b is a submodular function on subsets of E.

From section 3.2.2, we get $B_b = \mathcal{P}_f$, where

$$f(A, B) = b(A) + b(E/B) - b(E).$$

Thus,

$$f(A, \phi) = b(A), \text{ for all } A \subseteq E.$$

If x is an extreme point of $\mathcal{P}_f$, then there exists a maximal chain :

$$S : S^1 \subset S^2 \subset \cdots \subset S^n,$$

such that

$$x_i = f(S^i) - f(S^{i-1}) \text{ for } i = 1, 2, \ldots, n. \tag{4.3.3}$$

Clearly, in this case $S^i = (S_1^i, \phi)$. Now putting the values of f in terms of b in the relation (4.3.3), we get the required characterization of a point to be extreme point of a base polyhedron given by Fujishige [26].

(3) Pseudomatroid Polyhedron.

Chandrasekaran and Kabadi [7], define the pseudomatroid polyhedron as

$$\mathcal{P}_b = \{x(A, B) \leq b(A, B) \text{ for all } (A, B) \in D(E)\},$$

where $b \rightarrow \Re$ is an integer valued, bisubmodular function on $D(E)$ satisfying the following conditions :

1. $b(\phi, \phi) = 0$.
2. $b(e_i, \phi) = 0$ or 1 for all $e_i \in E$.
3. $\{A \subseteq C; \ D \subseteq B; \ (A, B), \ (C, D) \in D(E)\} \Rightarrow b(A, B) \leq b(C, D)$.

Here b is the rank function of the pseudomatroid $(E, \mathcal{F})$, represented by $\mathcal{P}_b$. Since this is a bisubmodular polyhedron, an extreme point x of $\mathcal{P}_b$ is characterized as in theorem (4.3.1).

Again b being the rank function of the pseudomatroid $(E, \mathcal{F})$, it follows that $b(S^i) - b(S^{i+1}) = 0$, if $e_i \in S_2^{i+1}\ \forall\, i$, and $b(S^i) - b(S^{i-1}) = 1$, for $e_i \in S_1^i\ \forall\, i$, therefore $x_i = 0$ or 1.

Let $A = \{e_i\ :\ x_i = 1\}$, then $b(A, \phi) = |A|$, implies $A \in \mathcal{F}$. Hence the pseudomatroid polyhedron is the convex hull of the characteristic vectors of the sets in $\mathcal{F}$.

(4) g-polymatroid.

From section 3.2.3, we know that a g-polymatroid $Q(p, b) = \mathcal{P}_f$, where

$$f(A, B) = b(A) + p(B),\ \forall\, (A, B) \in D(E).$$

Therefore,

$$f(A, \phi) = b(A) \text{ and } f(\phi, B) = p(B).$$

Now, $x \in \mathcal{P}_f$ will be an extreme point of $\mathcal{P}_f$, if there exists a maximal chain

$$S:\ S^1 \subset S^2 \subset \cdots \subset S^n,$$

such that

$$x_i = \begin{cases} f(S^i) - f(S^{i-1}) & \text{if } e_i \in S_1^i \\ f(S^{i-1}) - f(S^i) & \text{if } e_i \in S_1^i. \end{cases} \tag{4.3.4}$$

In this case

$$S^i = \begin{cases} (S_1^i, \phi) & \text{if } i \in F_q \\ (\phi, S_2^i) & \text{if } i \notin F_q. \end{cases}$$

Where F_q is an index set, with $|F_q| = q$.

Order the elements e_i such that $i \in F_q$ according to S_1^n and the remaining $(n-q)$ elements according to the reverse order of S_2^n, and putting the values of f in terms of b and p in the relation (4.3.4), we get

$$x_i = \begin{cases} b(S_1^i) - b(S_1^{i-1}) & \text{if } i \in F_q \\ p(G^{i+1}) - p(G^i) & \text{if } i \notin F_q \end{cases} \tag{4.3.5}$$

where $G^i = (e_i, e_{i+1}, \ldots, e_n)$.

This coincides with the result obtained by Hassin in [30].

(5) Perfectly Matchable Subgraph polytope.

We know that it is a pseudomatroid polyhedron, or more precisely a matching pseudomatroid polyhedron. So each extreme point of this polytope is a perfectly matchable subgraph of a graph $G = (V, E)$. And if the corresponding pseudomatroid is $(V, \mathcal{F})$, then this polyhedron is the convex hull of the characteristic vectors of the sets in $\mathcal{F}$.

(6) Degree Sequence Polytope.

If x is an extreme point of a degree sequence polytope then from section 3.4.4, there exists a chain

$$S : S^1 \subset S^2 \subset \cdots \subset S^n,$$

of the node set V, such that

$$x_i = \begin{cases} f(S^i) - f(S^{i-1}) & \text{if } i \in S_1^i \\ f(S^{i-1}) - f(S^i) & \text{if } i \in S_2^i \end{cases}$$

where $\{i\} = S^i / S^{i-1}$.

We have

$$f(S^i) = f(S_1^i, S_2^i) = |S_1^i|(n - 1 - |S_2^i|).$$

So

$$\begin{aligned} x_i &= f(S_1^i, S_2^i) - f(S_1^{i-1}, S_2^{i-1}) \qquad \text{if } i \in S_1^i \\ &= |S_1^i|(n-1-|S_2^i|) - |S_1^{i-1}|(n-1-|S_2^{i-1}|) \\ &= (n-1-|S_2^i|)(|S_1^i| - |S_1^{i-1}) \qquad \text{since } [|S_2^i| = |S_2^{i-1}|] \\ &= (n-1-|S_2^i|), \end{aligned} \tag{4.3.6}$$

or

$$\begin{aligned} x_i &= f(S_1^{i-1}, S_2^{i-1}) - f(S_1^i, S_2^i) \qquad \text{if } i \in S_2^i \\ &= |S_1^{i-1}|(n-1-|S_2^{i-1}|) - |S_1^i|(n-1-|S_2^i|) \\ &= |S_1^{i-1}|(n-1-|S_2^{i-1}| - n + 1 + |S_2^i|) \qquad \text{since } [|S_1^{i-1}| = |S_1^i|] \\ &= |S_1^{i-1}|. \end{aligned} \tag{4.3.7}$$

For nodes in S_1^n, the degrees are non-increasing and for nodes in S_2^n, the degrees are non-decreasing. With little more work it will be possible to show that x represents the degree sequence of a threshold graph, where S_1^n is the clique K and S_2^n is the stable set I.

To show the converse, let $(d_1, d_2, \ldots, d_n)$ be the degree sequences of a threshold graph on the node set $V = K \cup I$, where K denotes the clique and I the stable set of the threshold graph. K and I are uniquely realisable.

Let $S_1^n = K$ and $S_2^n = I$, arrange d_i's in non-decreasing order. Let i_1 be the first index for which $d_{i_1-1} < d_{i_1}$.
Define

$$S_1^1 = \{1\},\ S_2^1 = \{\phi\},\ S_1^2 = \{1,2\},\ S_2^2 = \{\phi\}, \ldots, S_1^{i_1-1} = \{1,2,\ldots,i_1-1\},\ S_2^{i_1-1} = \{\phi\},$$

and

$$S^{i_1} = S^{i_1-1} = \{1,2,\ldots i_1-1\},\ S_2^{i_1} = \{i_1\}.$$

Let i_2 be the next first index for which $d_{i_2-1} < d_{i_2}$.
Define

$$S_1^{i_1+1} = S_1^{i_1+2} = \cdots = S_1^{i_2-1} = S_1^{i_1-1},$$

and

$$S_2^{i_1+1} = \{i_1, i_1+1\}, \ldots, S_2^{i_2-1} = \{i_1, i_1+1, \ldots, i_2-1\},$$

and

$$S_1^{i_2} = \{1,2,\ldots,i_1-1,i_2\},\ S_2^{i_2} = \{i_1, i_1+1, \ldots, i_2-1\}.$$

We continue with this process till a full-length chain

$$S : S^1 \subset S^2 \subset \cdots \subset S^n,$$

is formed. This chain will generate the point $(d_1, d_2, \ldots, d_n)$, where

$$d_i = \begin{cases} f(S^i) - f(S^{i-1}) & \text{if } i \in K \\ f(S^{i-1}) - f(S^i) & \text{if } i \in I. \end{cases}$$

4.4 Characterization of Adjacency on $\mathcal{P}_f$

Adjacency of extreme points on the submodular polyhedron has been characterized in [52] and [54]. We generalise these results to define adjacency of extreme points on the bisubmodular polyhedron $\mathcal{P}_f$. Existing results for special bisubmodular polyhedra can be obtained from these general results.

Definition 4.4.1 Two extreme points x and $y \in \mathcal{P}_f$ are **adjacent** if and only if the collection of tight constraints with respect to x and y both has rank $(n-1)$.

4.4.1 Necessity

Here we establish necessary conditions for two extreme points of $\mathcal{P}_f$ to be adjacent.

Theorem 4.4.1 If x and y are adjacent extreme points of $\mathcal{P}_f$, then either $|\Delta(x,y)| = 1$ or $|\Delta(x,y)| = 2$.

Proof. Since x and y are given to be adjacent on $\mathcal{P}_f$, the sets of tight constraints with respect to x and y must have $(n-1)$ linearly independent constraints common and these $(n-1)$ constraints must define a 1-dimensional face, that is an edge of $\mathcal{P}_f$.

By lemma (4.2.2), the sets tight with respect to both x and y form a lattice, say $\mathcal{D}_3$ of rank $(n-1)$ and $\dim F(\mathcal{D}_3) = 1$ and $F^0(\mathcal{D}_3) \neq 0$.

Also $\mathcal{D}_3$ is a sublattice of a lattice $\mathcal{D}_1$ of tight sets with respect to x and a sublattice of lattice $\mathcal{D}_2$ of tight sets with respect to y.

This implies that there exist maximal chains $S : S^1 \subset S^2 \subset \cdots \subset S^n$ in $\mathcal{D}_1$ and $T : T^1 \subset T^2 \subset \cdots \subset T^n$ in $\mathcal{D}_2$ such that,
either

case (i) $T^i = S^i, \quad i = 1, 2, \ldots, (n-1)$

and $T^n = (S_1^n / e_n, S_2^n + e_n)$, if e_n is a forward element of S^n

$= (S_1^n + e_n, S_2^n/e_n)$, if e_n is a backward element of S^n.

Or

case (ii) $\quad T^i = S^i \; i = 1, 2, \ldots, n, i \neq l$ for some $l \in (1, (n-1))$,

and $\quad T^l = (S^{l+1}/e_l)$.

For case - (i),

$$y_i = x_i, \text{ for } i = 1, 2, \ldots, n-1.$$

and since x and y are distinct points of $\mathcal{P}_f$, $x_n \neq y_n$. This implies that $|\Delta(x, y)| = 1$.

For case - (ii), let us assume that e_l and e_{l+1} both are forward elements in S. Then

$$\begin{aligned} x_l &= f(S^l) - f(S^{l-1}) \\ \text{and } x_{l+1} &= f(S^{l+1}) - f(S^l) \\ \text{also } y_l &= f(S^{l+1}/e_l) - f(S^l) \\ \text{and } y_{l+1} &= f(S^{l+1}) - f(S^{l+1}/e_l). \end{aligned}$$

Since $x \neq y$, it follows from above that $f(S^l) \neq f(S^{l+1}/e_l)$ and in this case $x_l \neq y_l$ and $x_{l+1} \neq y_{l+1}$. Therefore $|\Delta(x, y)| = 2$, and T differs from S in the order of e_l and e_{l+1}. Other cases can be handled in a similar manner. □

4.4.2 Sufficiency

We now obtain sufficient conditions for two extreme points x and $y \in \mathcal{P}_f$ to be adjacent. Let $S : S^1 \subset S^2 \subset \cdots \subset S^n$ be a chain associated with x.

Theorem 4.4.2 If x and y are extreme points of $\mathcal{P}_f$ with $|\Delta(x, y)| = 1$, where $x_n \neq y_n$ then x and y are adjacent, and

$$y_n = \begin{cases} -x_n + \delta, & \text{if } e_n \text{ is a forward element in } S^n \\ -x_n - \delta, & \text{if } e_n \text{ is a backward element in } S^n. \end{cases}$$

Where

$$\delta = \begin{cases} f(S^n) - f(S_1^n/e_n, S_2^n + e_n), & \text{if } e_n \text{ is a forward element of } S \\ f(S^n) - f(S_1^n + e_n, S_2^n/e_n), & \text{if } e_n \text{ is a backward element of } S. \end{cases}$$

Proof. Since $x_i = y_i$, for all $i \neq n$, implies

$$\begin{aligned}
x(S^1) &= f(S^1) = y(S^1) \\
x(S^2) &= f(S^2) = y(S^2) \\
x(S^3) &= f(S^3) = y(S^3) \\
&\vdots \\
x(S^{n-1}) &= f(S^{n-1}) = y(S^{n-1}).
\end{aligned}$$

So we have $(n-1)$ linearly independent common tight constraints with respect to x as well as y, hence x and y are adjacent.

Given that y is an extreme point of $\mathcal{P}_f$, there exists at least one chain say T of n tight disets with respect to y. Since $S^1 \subset S^2 \subset \cdots \subset S^{n-1}$ already form a chain of length $(n-1)$, the n$^{\text{th}}$ set T^n can differ from S^n in e_n only. Let

$$T^n = \begin{cases} (S_1^n/e_n, S_2^n + e_n) & \text{if } e_n \text{ is a forward element of } S^n \\ (S_1^n + e_n, S_2^n/e_n) & \text{if } e_n \text{ is a backward element of } S^n \end{cases}$$

and $T : T^1 \subset T^2 \subset \cdots \subset T^n$, where $T^i = S^i$, $i = 1, 2, \ldots, (n-1)$, will form a chain of n-tight disets with respect to y.

Consider the case, when e_n is forward in S^n. Then $T^n = S^{n-1} \cup (\phi, e_n)$, and

$$\begin{aligned}
y_n &= -f(T^n) + f(T^{n-1}) \\
&= f(S^{n-1}) - f(T^n) \\
&= f(S^{n-1}) - f(S^n) + f(S^n) - f(T^n) \\
&= -x_n + f(S^n) - f(T^n) \\
&= -x_n + \delta.
\end{aligned}$$

In case e_n is a backward element in S^n, then

$$\begin{aligned}
y_n &= f(T^n) - f(T^{n-1}) \\
&= -f(S^{n-1}) + f(T^n) \\
&= -f(S^{n-1}) + f(S^n) - f(S^n) + f(T^n) \\
&= -x_n - f(S^n) + f(T^n) \\
&= -x_n - \delta. \ \square
\end{aligned}$$

Theorem 4.4.3 If x and y are two distinct extreme points of $\mathcal{P}_f$, and $\Delta(x,y) = \{e_l, e_{l+1}\}$, for some $l \in (1, n-1)$, where the ordering of the elements of $\mathcal{P}_f$ is with respect to some chain S of tight disets with respect to x, and T is a chain non-cancelling with S, with respect to y such that $S^i = T^i$ $i = 1, 2, \ldots n$, $i \neq l$ then x and y are adjacent and

$$\begin{array}{llll}
y_l = x_l + \delta, & y_{l+1} = x_{l+1} - \delta, & \text{if } \{e_l, e_{l+1}\} \text{ are both forward elements of } S \\
y_l = x_l - \delta, & y_{l+1} = x_{l+1} + \delta, & \text{if } \{e_l, e_{l+1}\} \text{ are both backward elements of } S \\
y_l = -x_l - \delta, & y_{l+1} = -x_{l+1} - \delta, & \text{if } e_l \text{ is a forward and } e_{l+1} \text{ is a backward element of } S \\
y_l = -x_l + \delta, & y_{l+1} = -x_{l+1} + \delta, & \text{if } e_l \text{ is a backward and } e_{l+1} \text{ is a forward element of } S.
\end{array}$$

Where $\delta = f(S^{l+1}/e_l) - f(S^l)$.

Proof. By the statement of the theorem, the ordering of T is

$$T = \{e_1, e_2, \ldots, e_{l-1}, e_{l+1}, e_l, e_{l+2} \ldots, e_n\}.$$

From this it follows that x and y satisfy the $(n-1)$ tight constraints with respect to the disets $S^1, S^2, \ldots S^{l-1}, S^{l+1}, \ldots, S^n$, which are linearly independent. Hence x and y are adjacent.

In case $\{e_l, e_{l+1}\}$ are both forward elements of S,

$$\begin{aligned}
y_l &= f(S^{l+1}/e_l) - f(S^{l-1}) \\
&= f(S^{l+1}/e_l) - f(S^l) + f(S^l) - f(S^{l-1}) \\
&= x_l + \delta,
\end{aligned}$$

and

$$\begin{aligned}
y_{l+1} &= f(S^{l+1}) - f(S^{l+1}/e_l) \\
&= f(S^{l+1}) - f(S^l) + f(S^l) - f(S^{l+1}/e_l) \\
&= x_{l+1} - \delta.
\end{aligned}$$

The remaining cases can be handled in the same way. □

Theorems (4.4.2) and (4.4.3) allow us to bound the member of extreme points adjacent to a given extreme point on a bisubmodular polyhedron.

From theorem (4.4.2), there can be at most $|E|$ adjacent extreme points, since the n^{th} position in a partial order with respect to the given extreme point may be occupied by any $e \in E$. And from theorem (4.4.3), for every distinct pair $e_i, e_j \in E$, there can be at most one adjacent extreme point, which differs from the given extreme point in the order of e_i, e_j.

The total number of such adjacent extreme points is therefore bounded by $\frac{(|E|-1)|E|}{2}$.

Thus an upper bound on the number of extreme points adjacent to a given extreme point is $(|E|^2 + |E|)/2$.

Going back to example (4.2.1), consider the adjacent points $(1,1)$ and $(-1,-1)$. Here $|\Delta(x,y)| = 2$.

$$S:\ \phi = S^0 \subset S^1 = (e_2, \phi) \subset S^2 = (e_2, e_1)$$

is a chain of tight sets with respect to $(1,1)$.

Construct the chain T by reversing the order of e_2 and e_1 in S. Therefore

$$T:\ \phi = T^0 \subset T^1 = (\phi, e_1) \subset T^2 = (e_2, e_1).$$

We see that T is the chain of tight sets with respect to $(-1,-1)$, and S and T have one tight set common between them.

If we consider the adjacent points $(1,1)$ and $(1,0)$, then $|\Delta(x,y)| = 1$ and

$$S:\ \phi = S^0 \subset S^1 = (e_1, \phi) \subset S^2 = (e_1\, e_1, \phi)$$

is a chain of tight sets with respect to $(1,1)$.

Construct the chain T, by making the last element of S, backward, so,

$$T:\ \phi = T^0 \subset T^1 = (e_1, \phi) \subset T^2 = (e_1, e_2)$$

and this gives the tight sets $x_1 = 1$, $x_1 - x_2 = 1$ with respect to $(1,0)$.

4.4.3 Adjacency on Different Kinds of Bisubmodular Polyhedra

In this section we characterize adjacency on special kinds of bisubmodular polyhedra. Using the sufficiency criteria given in theorems (4.4.2) and (4.4.3). We assume that these polyhedra are all full dimensional.

(1) Ditroid Polyhedron $\mathcal{P}_h$.

If x and y are two extreme points of $\mathcal{P}_h$, then x and y are the characteristic vectors of independent disets say I^1 and I^2 of the corresponding ditroid $D = (E, \mathcal{I})$.

If x and y are adjacent on $\mathcal{P}_h$ and $|\Delta(x,y)| = 1$, it follows from theorem (4.4.2), that in case $x_n = 1$, $y_n = -1$ or 0, and in case $x_n = -1$, $y_n = 1$ or 0.

And if $|\Delta(x,y)| = 2$, then from theorem (4.4.3), $\delta = f(S^{l+1}/e_l) - f(S^l) \neq 0$ and if $x_l = 1$, then $y_l = 0$, i.e., $\delta = -1$. This implies that $x_{l+1} = 0$ and $y_{l+1} = 1$. Therefore, if $e_l \in I^1$ as a forward element and $e_{l+1} \notin I^1$, then I^2 contains e_{l+1} and e_l does not belongs to it.

(2) Degree Sequence Polytope D_n.

If x and y are extreme points on D_n, then since they represent degree sequences of two graphs on the same node set V, $|\Delta(x,y)| \neq 1$.

Also as seen in section 4.3.1 (6) of this thesis, x must be a degree sequence of a threshold graph $G(V, E_1)$ and y must be a degree sequence of a threshold graph $G(V, E_2)$.

It can be easily verified that the various characterizations of adjacency given in lemmas 3.2, 3.3 and 3.4 of [41], all satisfy conditions of our theorem (4.4.3) and that our theorems (4.4.1) and (4.4.3) reduce to the condition $|E_1 \oplus E_2| = 1$.

(3) Base Polyhedron $\mathcal{B}_b$.

Since a fully dimensional base polyhedron has dimension $(|E| - 1)$, the statements of theorems (4.4.1), (4.4.2) and (4.4.3) will need to be modified.

In any case, since we are not dealing with directed sets, second part of theorem (4.4.1)

and theorem (4.4.3) will be applicable, and it can be shown that in this case these two theorems reduce to the characterization given by Fujishige [26] for two extreme points of a base polyhedron to be adjacent.

Adjacency criteria on polymatroid polyhedron and g-polymatroids are obtainable from the criteria on $\mathcal{B}_b$ and hence these results are also special cases of theorems (4.4.1), (4.4.2) and (4.4.3).

(4) Pseudomatroid Polyhedron $\mathcal{P}_b$.

As was mentioned earlier, the pseudomatroid polyhedron is the convex hull of the characteristic vectors of the independent sets of the corresponding pseudomatroid. Let $x,\ y \in \mathcal{P}_b$ be extreme points and let $x = \chi(A)$, $y = \chi(B)$, $A,\ B \in F$ (the set of independent sets of the pseudomatroid).

Theorem (4.4.2) implies that A and B are adjacent independent sets if $A \subseteq B$ and $|A| = |B| - 1$. Theorem (4.4.3) implies that A and B are adjacent independent sets on $\mathcal{P}_b$ if $|A| = |B|$ and $A = (B + j)/k$ for some $j \in A/B$ and $k \in B/A$.

The perfectly matchable subgraph polytope $\mathcal{P}_\rho$ is a pseudomatroid polyhedron and hence the characterization of adjacency on $\mathcal{P}_\rho$ is obtainable from theorems (4.4.1), (4.4.2) and (4.4.3). Which reduce to the characterization of adjacency by Balas and Pulleyblank [1] on $\mathcal{P}_\rho$.

4.5 Separation Theorem

In this section we prove a separation theorem for the bisubmodular polyhedron $\mathcal{P}_f$. Separation theorems for polymatroids [2] and for the delta-matroid polyhedron are special cases of this theorem. It is also shown that the bisubmodular function minimization and the separation problem for the bisubmodular polyhedron $\mathcal{P}_f$ are equivalent.

Definition 4.5.1 For $x,\ y \in \Re^n$ we say $y \prec x$ if and only if $|y_e| \leq |x_e|$ for all $e \in E$.

We first prove a min-max theorem for $\mathcal{P}_f$.

Theorem 4.5.1 Let f be bisubmodular on $D(E)$ such that $f(\phi,\phi) = 0$ and $x \in \Re^n$. Then

$$\max\{y(X,Y) : y \prec x,\, y \in \mathcal{P}_f\} = \min\{f(A,B)+x(X/A,Y/B);\, \forall (A,B) \subseteq (X,Y)\}, \quad (4.5.1)$$

where

$$X = \{e \in E \,:\, x_e \geq 0\} \text{ and } Y = \{e \in E \,:\, x_e < 0\}.$$

Moreover, if f is integer-valued, there exists a maximizing y each of whose component is an integer combination of elements of $\{x_e \,:\, e \in E\} \cup \{1\}$.

Proof. Let $(A_0, B_0) \subseteq (X,Y)$ be the minimizing diset for the right hand side of (4.5.1).

$$\begin{aligned} \text{since } y \in \mathcal{P}_f \quad &\Rightarrow \quad y(A_0,B_0) \leq f(A_0,B_0) \\ \text{and } y \prec x \quad &\Rightarrow \quad y(X/A_0,Y/B_0) \leq x(X/A_0,Y/B_0). \end{aligned}$$

These two together imply that

$$y(X,Y) \leq f(A_0,B_0) + x(X/A_0,Y/B_0),\ \forall y \in \mathcal{P}_f \text{ and } y \prec x, \quad (4.5.2)$$

therefore, right hand side value $\geq$ left hand side value.

Now consider any $y \in \mathcal{P}_f$, such that $y \prec x$. If for some $e \in X$, $y_e < x_e$, increase y_e by

$$\min\{x_e - y_e,\, \min\{f(A,B) - y(A,B) \,:\, (A,B) \subseteq (X,Y)\}, e \in A\},$$

and the new y will still be feasible. Thus for every $e \in X$, such that $y_e < x_e$ and y_e cannot be increased any further, there exists $(X_1,X_2) \subseteq (X,Y)$ such that $y(X_1,X_2) = f(X_1,X_2)$, and $e \in X_1$.

Let (A,B) denote the union of all such sets. Then $y(A,B) = f(A,B)$ and for all $e \in X$ such that $y_e < x_e$, $e \in A$.

Similarly, if $e \in Y$ and $y_e > x_e$, y_e can be reduced by adding

$$\max\{x_e - y_e,\, \max\{y(Y_1,Y_2) - f(Y_1,Y_2) \,:\, (Y_1,Y_2) \subseteq (X,Y)\}, e \in Y_2\},$$

and the new y will still be feasible. Thus for every $e \in Y$ such that $y_e > x_e$ and y_e cannot be reduced any further, there exists $(Y_1, Y_2) \subseteq (X, Y)$, for which $y(Y_1, Y_2) = f(Y_1, Y_2)$, and $e \in Y_2$.

Let (C, D) denote the union of all such sets. Then $y(C, D) = f(C, D)$ and for all $e \in Y$ such that $y_e > x_e$, $e \in D$.

By definition, (A, B) and (C, D) are non-cancelling and

$$y((A,B) \cup (C,D)) = f((A,B) \cup (C,D)) = f(A \cup C, B \cup D).$$

Also

$$\begin{aligned} y(X,Y) &= y(A \cup C, B \cup D) + y(X/(A \cup C), Y/(B \cup D)) \\ &= f(A \cup C, B \cup D) + x(X/(A \cup C), Y/(B \cup D)). \end{aligned}$$

This implies that the left hand side maximum value is greater than or equal to the right hand side minimum value. Hence the equality of the left hand and right hand side values.

We see from the equality for $y(X, Y)$, that in case f is integer valued, y is an integer combination of $\{x_e : e \in E\} \cup \{1\}$. Hence the theorem. □

An immediate consequence of this theorem is the solution of the membership problem for $\mathcal{P}_f$.

Membership problem.

Given $x \in \Re^n$ and a bisubmodular polyhedron $\mathcal{P}_f$, does $x \in \mathcal{P}_f$?

Let y_0 be the maximizing vector of the left hand side of (4.5.1), where (X, Y) is defined as in theorem (4.5.1), and (A_0, B_0) the minimizing diset of right hand side of (4.5.1).

Then

$$y_0(X,Y) = f(A_0, B_0) + x(X/A_0, Y/B_0).$$

In case

$$f(A_0, B_0) + x(X/A_0, Y/B_0) < x(X,Y) = \sum_{i=1}^{n} |x_i|,$$

$$\Rightarrow \qquad f(A_0, B_0) < x(A_0, B_0).$$

Hence $\quad x \notin \mathcal{P}_f$.

But if $y_0(X,Y) = x(X,Y)$, this implies $y_0 = x$, since $y_0 \prec x$. Therefore $x \in \mathcal{P}_f$.

In case $\mathcal{P}_f$ is the pseudomatroid polyhedron, the min-max theorem will require $x \in \Re^n_+$ and in this case $X = E$, $Y = \phi$ and our theorem will reduce to the min-max theorem for pseudomatroid polyhedron proved in [12].

Bisubmodular function minimization.

$$\text{BFMP}: \qquad \min_{X \in D(E)} f(X),$$

where f is bisubmodular on $D(E)$.

Bisubmodular separation problem is :

BSP : Given $x^* \in \Re^n$, does $x \in \mathcal{P}_f$? If not, find $X \in D(E)$ such that the violation of $x^*(X) \leq f(X)$ is maximized.

Define another diset function g on $D(E)$ by

$$g(X) = f(X) + x^*(X).$$

It can be easily shown that g is bisubmodular.

Theorem 4.5.2 The following two statements are equivalent.

1. X^* is the minimizing diset for BFMP.
2. For the separation problem BSP, with respect to $\mathcal{P}_g$ and $x^* \in \Re^n$, $x(X^*) \geq g(X^*)$ is a most violated inequality.

Proof. Let X^* be an optimal solution for BFMP.

$$\begin{aligned} f(X^*) &\leq f(Y), \text{ for all } Y \in D(E) \\ \text{iff } g(X^*) - x^*(X) &\leq g(Y) - x^*(Y). \end{aligned}$$

The first inequality is equivalent to (1) and second inequality to (2). □

In [2] a finite algorithm though not polynomial, for determining membership in a polymatroid is given. Since it has been shown for the bisubmodular polyhedron $\mathcal{P}_f$, that the number of adjacent extreme points to a given extreme point is of $\mathcal{O}(|E|^2)$ and theorems (4.4.2) and (4.4.3) give formulae for obtaining these adjacent extreme points, it should be possible to extend the algorithm in [2] to the bisubmodular polyhedron $\mathcal{P}_f$, provided a starting feasible solution to the left hand side of (4.5.1) can be easily obtained. But obtaining a starting feasible solution appears to be as difficult as the membership problem.

In [43], Qi has defined the Lovasz extension $\hat{f}$ of a bisubmodular function f on $D(E)$, with $f(\phi, \phi) = 0$. He then shows that,

$$\min\{f(X) \ : \ X \in D(E)\} = \min\{\hat{f}(u) \ : \ u \in [-1, +1]^{|E|}\}.$$

Algorithm of Judin and Nemirorskii can be applied to the right hand side problem to obtain the minimizing $X \in D(E)$ in polynomial time. However, computationally the algorithm is not very attractive.

In [50], a pseudo-polynomial algorithm for deciding membership in submodular polyhedra is given. In fact the author claims, that the algorithm is applicable to all other polyhedra over which optimization may be done easily. Since the greedy algorithm applies to the bisubmodular polyhedra this algorithm should work. Hence a pseudo-polynomial algorithm for minimizing a bisubmodular function exists.

A special case of the membership problem for the bisubmodular polyhedron is the membership problem for the ditroid polyhedron. Since adjacency on the ditroid polyhedron is fully characterized an attempt was made to extend Cunningham's algorithm in [9] for the ditroid polyhedron membership. Here also, obtaining a starting feasible y as a convex combination of the extreme points of the ditroid polyhedron does not seem to be easy.

4.6 Conclusions

It should be possible to extend much more of the matroid theory to ditroids. Also because of lack of time not much work could be done on greedy systems, which have been introduced in this thesis.

Apart from an ellipsoidal algorithm for minimizing a submodular function, the other algorithm are either just finite [2] or pseudo polynomial. Here in this thesis, we have only indicated that the pseudo polynomial algorithm in [50] could be extended for membership problem for a bisubmodular polyhedron and hence for minimizing a bisubmodular function. It should be possible to exploit the polyhedral structure of the bisubmodular polyhedron and construct a polynomial time algorithm for the corresponding membership problem

Construction of monotone paths on the bisubmodular polyhedron will be a worth considering problem.

Another problem worth mentioning is, what kind of single constraint can be added to the LPP with respect to a bisubmodular polyhedron so that integrality property and also the greediness property is maintained.

REFERENCES

1. BALAS, E., PULLEYBLANK, W. R. : The Perfectly Matchable Subgraph Polytope of an Arbitrary Graph, *Combinatorica*, 9 (1989) 321-337.

2. BIXBY, R. E., CUNNINGHAM, W. H., TOPKIS, D. M. : The Partial Order of Polymatroid Extreme Point, *Mathematics of Operations Research*, 3 (1984) 367-378.

3. BLAND, R. G., VERGNAS, M. L. : Orientablity of Matroids, *Journal of Combinatorial Theory*, B, 24 (1978) 94-123.

4. BOUCHET, A. : Greedy Algorithm & Symmetric Matroids, *Mathematical Programming*, 38 (1987) 147-159.

5. BOUCHET, A. : Representability of Δ - Matroids, *Colloquia Societatis Janos Bolyai*, 52 (1988) 167-182.

6. BOUCHET, A., CUNNINGHAM, W. H. : Delta Matroids, Jump Systems & Bisubmodular Polyhedra. preprint.

7. CHANDRASEKARAN, R., KABADI, S. N. : Pseudomatroids, *Discrete Mathematics*, 71 (1988) 205-217.

8. CONFORTI, M., LAURENT, M. : A Characterization of Matroidal Systems of Inequalities, *Discrete Mathematics*, 2 (1988) 165-171.

9. CUNNINGHAM, W. H. : Testing membership in matroid polyhedra, *Journal of Combinatorial Theory*, Series-B 36 (1984) 161-188.

10. CUNNINGHAM, W. H. : On Submodular Function Minimization, *Networks*, 5 (1985) 185-192.

11. CUNNINGHAM, W. H., FRANK, A. : A Primal-Dual Algorithim for Submodular Flows, *Mathematics of Operations Research*, 10 (1985) 251-261.

12. CUNNINGHAM, W. H., KROTKI, J. G. : b-matching Degree Sequences Polyhedra, Report 88535- OR, Institut fur O.R, Bonn, 1988.

13. CUNNINGHAM, W. H., KROTKI, J. G. : A Separation Algorithim for The Matchable Set Polytope, Preprint.

14. DUNSTAN, F. D. J., WELSH, D. J. A. : A Greedy Algorithim Solving a Certain Class of Linear Programmes, *Mathematical Programming*, 5 (1973) 338-353.

15. EDMONDS, J. : Submodular Functions, Matroid and Certain Polyhedra, in R. Guy, H. Hanani, N. Saner and J. Schonheim, eds. *Combinatorial Structures and Their Applications*, (Gordon and Breach, New York, 1970), 69-87.

16. EDMONDS, J., GILES, R. : A Min-Max Relation on Submodular Functions on Graph, *Annals of Discrete Mathematics*, 1 (1977) 185-204.

17. EDMONDS, J., GILES, R. : Total Dual Integrality of Linear Inequility System, in W. R. Pulleyblank ed. Progress in Combinatorial Optimization (Academic Press, Toronto, Ontario, 1984)

18. EDMONDS, J., JOHNSON, E. L. : Matching : A Well-Solved Class of Integer Linear Programs, in R. Guy, H. Hanani, N. Saner and J. Schonheim, eds. *Combinatorial Structures and Their Applications*, (Gordon and Breach, New York, 1970), 88-92.

19. FRANK, A. : Submodular Flows, in W. R. Pulleyblank ed. Progress in Combinatorial Optimization (Academic Press, Toronto, Ontario, 1984) 147-166.

20. FRANK, A., : Kernal System of Directed Graphs, *Acta Scientiarum Mathematicarum*, 41 (1979) 63-76.

21. FRANK, A.,: An Algorithim For Submodular Functions on Graphs, *Annals of Discrete Mathematics*, 16 (1982) 97-120.

22. FRANK, A. : Generalised Polymatroids, in A. Hajnal et. al.eds. Finite & and infinite sets (North Holland, Amsterdam- New York, 1984, 285-294.

23. FRANK, A. : Finding Feasible Vectors of Edmonds-Giles Polyhedra, *Journal of Combinatorial Theory*, (B), 36 (1984) 221-239.

24. FRANK, A., TARDOS, E. : Generalised Polymatroids & Submodular Flows, *Mathematical Programming*, 42 (1988) 489-563.

25. FUJISHIGE, S.: Principal Structures of Submodular Systems, *Discrete Applied Mathematics*, 7(1984) 105-109.

26. FUJISHIGE, S. :A Characterization of Faces of The Base Polyhedron Associated With a Submodular System, *Journal of the O.R Society of Japan*, 27 (1984) 112-128.

27. FUJISHIGE, S.: Submodular Systems and Related Topics, in B. Korte and K. Ritter eds. *Mathematical Programming Study*, 22 (1984) 113-131.

28. FUJISHIGE, S. :Submodular Function & Optimization. North-Holland, Amsterdam, 1991.

29. FUJISHIGE, S., TOMIZAWA, N. : A Note on Submodular Function on Distributive Lattices, *Journal of the O.R Society of Japan*, 26 (1983) 309-318.

30. HASSIN, R. : Minimum Cost Flow With Set Constraints, *Networks*, 12 (1982) 1-22.

31. HOUSMANN, D. : Adjacency on Polytope in Combinatorial Optimization, *Mathematical System in Economics*, 49, Oegeschalager, Gunn and Hain (Cambridge, Massachusetts, 1979).

32. HOUSMANN, D., KORTKI, B. : Colouring Criteria for Adjacency on 0 - 1 Polyhedra, *Mathematical Programming Study*, 8 (1978) 106-127.

33. KABADI, S. N., CHANDRASEKARAN, R. : On Totally Dual Integral System, *Discrete Applied Mathematics*, 26 (1990) 87-104.

34. KOREN, M. : Extreme Degree Sequences of Simple Graphs, *Journal of Combinatorial Theory*, B 15 (1973) 213-224.

35. LAWLER, E. L., MARTEL, C. U. : Computing Maximal 'Polymatroidal' Network Flows, *Mathematics of Operations Research*, 7 (1982) 369-374.

36. LOVASZ, L. : Matriod Matching and Some Applications, *Journal of Combinatorial Theory*, B, 28 (1980) 208-236.

37. LOVASZ, L. : Submodular Function & Convexity, in A. Bachem, M. Grotschel & B. Korte, eds. *Mathematical Programming*, The State of The Art (Springer- Verlag, Berlin-NewYork, 1983), 235-257.

38. McDIARMID, C. : Blocking, Antiblocking and Pairs of Matroids and Polymatroids, *Journal of Combinatorial Theory* (B), 25 (1978) 313-325.

39. NEMHAUSER, G. L., WOLSEY, L. A. : Integer & Combinatorial Optimization, John Wiley & Sons, New York, 1988.

40. PAPADAMITROU, C. H., STEIGLITZ, K. : Combinatorial Optimization : Algorithims & Complexity, Prentice-Hall Inc. New Jersey, 1982.

41. PELED, U. N., SRINIVASAN, M. K. : The Polytope of Degree Sequences, *Liner Algebra and Its Applications*, 114/115 (1989) 349-377.

42. QI, LIQUN. : Odd submodular functions, Dilworth-functions & discrete convex function, *Mathematics of Operations Research*, 13 (1988) 435-446.

43. QI, LIQUN. : Directed Submodularity,Ditroids & Directed Submodular Flows, *Mathematical Programming*, 42 (1988) 579-599.

44. QI, LIQUN. : Bisubmodular Function, Core discussion paper No-8901, 1989.

45. SCHRIJVER, A. : Matroids and Linking System, *Journal of Combinatorial Theory*, B, 26 (1979) 349-369.

46. SCHRIJVER, A. : Proving Total Dual Integrality With Cross-Free Families - A General Framework, *Mathematical Programming*, 29 (1984) 15-27.

47. SCHRIJVER, A. : Total Dual Integrality From Directed Graphs, Crossing Families & Sub and Supermodular Functions, in W.R. Pulleyblank ed. Progress in Combinatorial Optimization, (Academic Press, Toronto, Ontario. 1984) 315-362.

48. SCHRIJVER, A. : Theory of Liner & Integer Programming, (Wiley, New York, 1986).

49. SHAPLEY, L. S. : Cores of Convex Games, *International Journal of Game Theory*, 1 (1971) 11-26.

50. SOHONI, M. : Membership in Submodular and Other Polyhedra, TR - 102 - 92.

51. TARDOS, E. : Generalized Matroids and Submodular Colourings, Report AE 19/83, University of Amsterdam.

52. TOPKIS, D. M. : Adjacency on Polymatroids, *Mathematical Programming* , 30 (1984) 229-237.

53. WELSH, D. J. A. : Matroid Theory, Academic Press, London, 1976.

54. ZHANG, S. : On Vertices Adjacency Characterization and Optimization of Polymatroid, Report No.8916/A, Erasmus University, Rotterdam.

A 121758

